Codes and Cryptography

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PART V

Outline

Block Codes

From now on, we assume:

- An r-ary code $\mathcal{C}$ of constant length n and size K
- A K-ary uniform source (all codewords are equally likely)
- An r-ary symmetric channel with (small) error probability α

In a **block code** a source stream is split into blocks (of the same size), and each block is encoded independently.

Minimum Distance and Error Correction Capacity

The **minimum distance** of $\mathcal{C}$ is defined as

$$d_{\mathcal{C}} = \min_{x,y \in \mathcal{C},\, x \neq y} d(x,y)$$

- The *nearest codeword* decoding rule always gives the right codeword if they occur at most $t = \lfloor \frac{d_{\mathcal{C}}-1}{2} \rfloor$ errors.
  t is called the **error correction capability** of $\mathcal{C}$.
- $\mathcal{C}$ can also correct up to $d_{\mathcal{C}} - 1$ erasures.

The **relative minimum distance** is defined as $\delta_{\mathcal{C}} = \frac{d_{\mathcal{C}}}{n}$, where n is the length of $\mathcal{C}$.

Singleton Bound

If we puncture $\mathcal{C}$ at some position $i \in \{1, \ldots, n\}$, and $d = d_{\mathcal{C}} > 1$, we obtain a new code $\mathcal{C}'$ of length $n' = n - 1$, minimum distance $d' \geq d - 1$ and size $|\mathcal{C}'| = |\mathcal{C}|$.

By puncturing $\mathcal{C}$ at $d - 1$ positions we similarly obtain another code of the same size and length $n - d + 1$.

Therefore,

$$|\mathcal{C}| \leq r^{n-d+1}$$

or equivalently, in terms of the code rate,

$$R = \frac{\log_r |\mathcal{C}|}{n} \leq 1 - \delta + \frac{1}{n}$$

MDS Codes

Codes achieving the equality in Singleton bound are called **Maximum Distance Separable** or **MDS**.

Examples of them are

- the trivial code, $\{0, \ldots, r-1\}^n$
- the repetition code, $\mathcal{C}_{rep} = \{0^n, \ldots, (r-1)^n\}$
- the "hyperplane"
  $\mathcal{C} = \{\mathbf{x} \in \{0, \ldots, r-1\}^n \mid x_1 + \ldots + x_n = 0\}$
  $|\mathcal{C}| = r^{n-1}$, $d_{\mathcal{C}} = 2$, it corrects up to 1 erasure, but no errors

The only MDS codes for $r = 2$ and d odd are the trivial and the repetition codes.

Sphere Packing (Hamming) Bound

Given any code $\mathcal{C}$ of length n and minimum distance $d > 1$, the spheres in $\{0, \ldots, r-1\}^n$ of radius $t = \lfloor \frac{d-1}{2} \rfloor$ centered at codewords of $\mathcal{C}$ are disjoint. Therefore,

$$|\mathcal{C}| \leq \frac{r^n}{1 + n(r-1) + \ldots + \binom{n}{t}(r-1)^t}$$

Codes achieving the equality are called **Perfect**.

If $\mathcal{C}$ is a binary MDS code, $1 + n + \ldots + \binom{n}{t} \leq \frac{2^n}{|\mathcal{C}|} = 2^{d-1}$

Using now $n \geq d \geq 2t + 1$,

$1 + n + \ldots + \binom{n}{t} \geq 1 + (2t+1) + \ldots + \binom{2t+1}{t} = 2^{2t}$

If $d \geq 3$ is odd, $d - 1 = 2t$, and then $n = d$, $|\mathcal{C}| = 2$, and $\mathcal{C}$ can only be the repetition code.

Plotkin Bound

For a random $X = (X_1, \ldots, X_n) \in \mathcal{C}$, define $p_{i,j} = \Pr(X_i = j)$.
Now, for two independent random codewords X, Y,

$$\mathrm{E}(d(X,Y)) = \sum_{i=1}^{n} \Pr(X_i \neq Y_i) = \sum_{i=1}^{n} \left(1 - \sum_{j=0}^{r-1} p_{i,j}^2\right) \leq \sum_{i=1}^{n} \left(1 - \frac{1}{r}\right)$$

where the last inequality comes from $\|\mathbf{v}\|_1^2 \leq r\|\mathbf{v}\|_2^2$, for $\mathbf{v} \in \mathbb{R}^r$.
Therefore,

$$d_{\mathcal{C}} \left(1 - \frac{1}{|\mathcal{C}|}\right) = d_{\mathcal{C}} \Pr(X \neq Y) \leq \mathrm{E}(d(X,Y)) \leq n \left(1 - \frac{1}{r}\right)$$

and for any $d_{\mathcal{C}} > \frac{r-1}{r} n$, $\qquad |\mathcal{C}| \leq \frac{r d_{\mathcal{C}}}{r d_{\mathcal{C}} - (r-1)n}$

Example

$\mathcal{C} = \{0000, 0122, 0211, 1012, 1101, 1220, 2021, 2110, 2202\}$

- $r = 3$, $n = 4$, $|\mathcal{C}| = 9$, $d = 3$
- $\mathcal{C}$ corrects one error and has a rate $R = 1/2$ (while $\mathcal{C}_{rep}$ with $n = 3$ also corrects one error, but at a rate $1/3$).
- $\mathcal{C}$ is MDS and perfect, since $|\mathcal{C}| = 3^{n-d+1}$ and $|\mathcal{C}| = \frac{3^n}{1+2n}$
- $\mathcal{C}$ satisfies the equality in Plotkin bound, since $|\mathcal{C}| = \frac{3d}{3d-2n}$

Actually, $\mathcal{C}$ can be seen as a vector subspace in $\mathbb{F}_3^4$ (so codewords can be added or multiplied by scalars in $\mathcal{C}$).

Furthermore, it can be seen as the ideal generated by the polynomial $X^2 + 2X + 2$ in the quotient ring $\mathbb{F}_3[X]/(X^4 + 1)$
e.g., the codeword 1012 corresponds to the polynomial $X^3 + X + 2 = (X + 1)(X^2 + 2X + 2)$

Maximal Codes

An r-ary code $\mathcal{C}$ of length n and minimum distance d is called **maximal** if it has the maximum possible size. We denote the size of a maximal code as $A_r(n, d)$.

Every of the previous bounds (Singleton, Hamming and Plotkin) are indeed upper bounds for $A_r(n, d)$. Namely

- $A_r(n, d) \leq r^{n-d+1}$
- $A_r(n, d) \leq \dfrac{r^n}{1 + n(r-1) + \ldots + \binom{n}{t}(r-1)^t}$
- if $d > \dfrac{r-1}{r} n$ then $A_r(n, d) \leq \dfrac{r d_{\mathcal{C}}}{r d_{\mathcal{C}} - (r-1)n}$

On the other hand, every constuction of a code gives a lower bound for $A_r(n, d)$. But there is a known non-constructive lower bound.

Gilbert-Varshamov Bound

Given a maximal r-ary code $\mathcal{C}$ of length n and minimum distance d, the spheres in $\{0, \ldots, r-1\}^n$ of radius $d-1$ centered at codewords of $\mathcal{C}$ must cover all the space. Otherwise, any $x \in \{0, \ldots, r-1\}^n$ outside all spheres can be added to $\mathcal{C}$ as a new codeword, without decreasing the minimum distance. Therefore,

$$A_r(n, d) \geq \frac{r^n}{1 + n(r-1) + \ldots + \binom{n}{d-1}(r-1)^{d-1}}$$

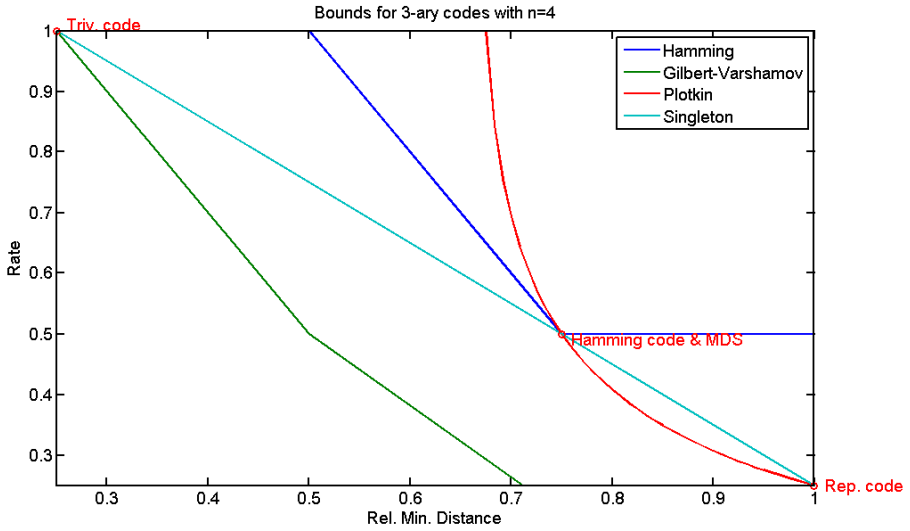
The **covering radius** of a code $\mathcal{C}$ is defined as

$$\rho_{\mathcal{C}} = \max_{x \in \{0,\ldots,r-1\}^n \setminus \mathcal{C}} d(x, \mathcal{C})$$

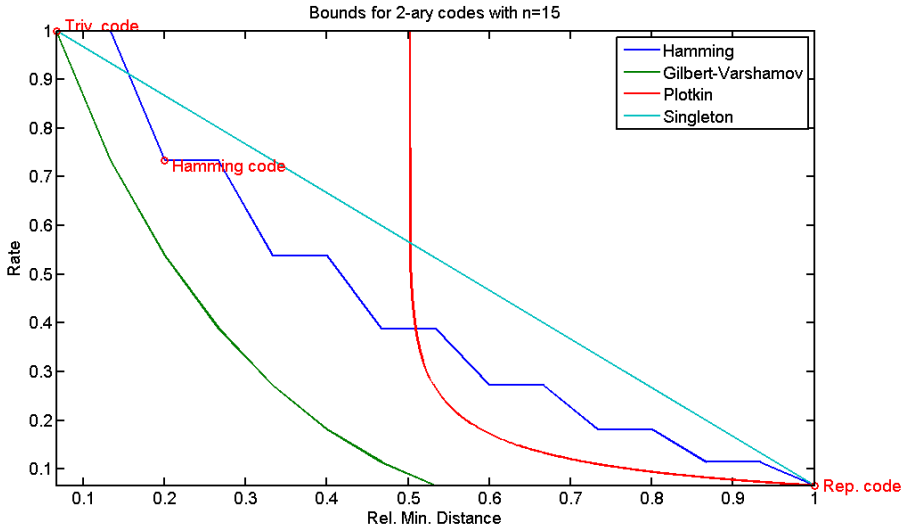
where

$$d(x, \mathcal{C}) = \min_{y \in \mathcal{C}} d(x, y)$$

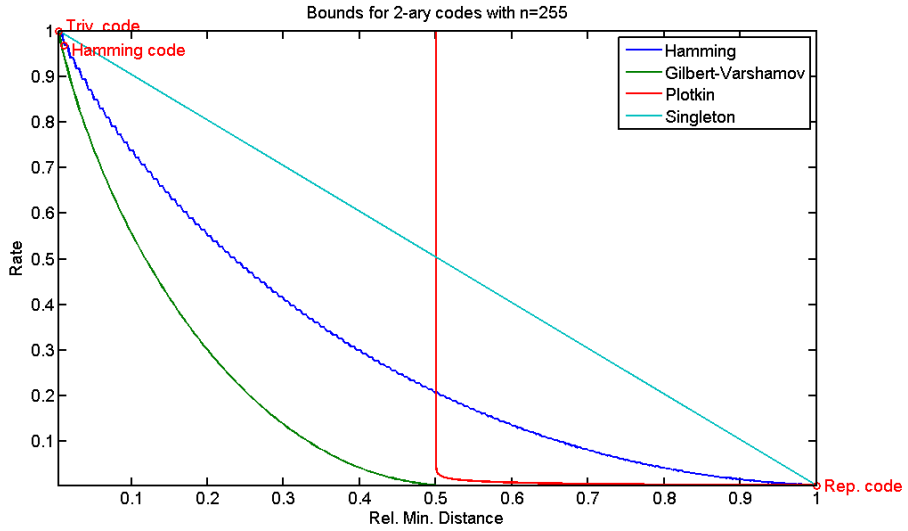
Plot of the Bounds (I)



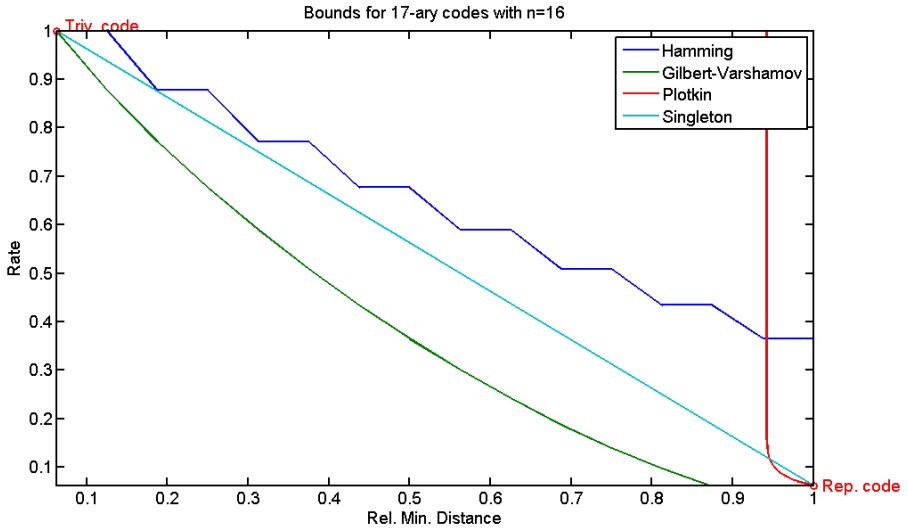
Plot of the Bounds (II)



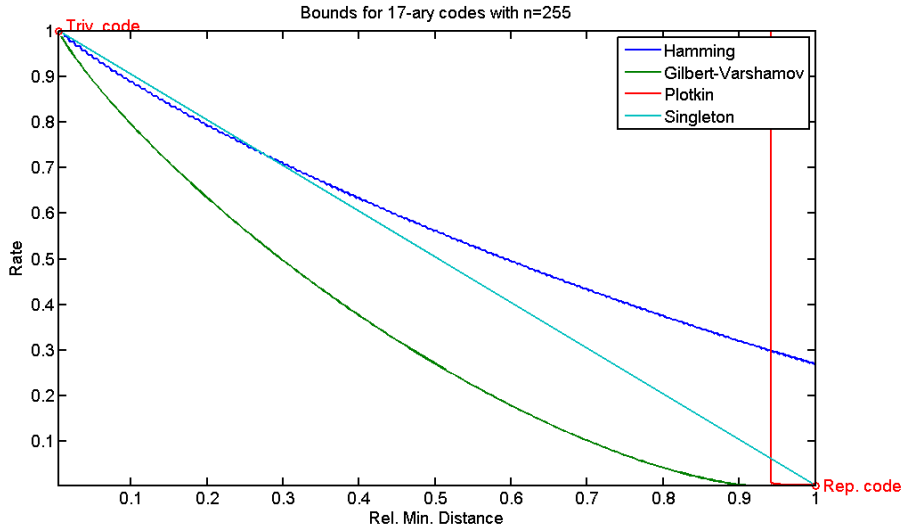
Plot of the Bounds (III)



Plot of the Bounds (IV)



Plot of the Bounds (V)



Outline

Prime Fields

The quotient ring $\mathbb{Z}/n\mathbb{Z}$ is a field if and only n is prime. We denote it as $GF(n)$ or $\mathbb{F}_n$. It is usually known as a **prime field**.

If n is a composite integer then $\mathbb{Z}/n\mathbb{Z}$ has zero divisors. On the other hand, Bezout's inequality guarantees the existence on inverses modulo n when n is prime.

The **characteristic** of a finite field $\mathbb{K}$, $\mathrm{char}(\mathbb{K})$, is the minimum positive integer p such that $p1_{\mathbb{K}} = 0_{\mathbb{K}}$. It is always a prime number. (Otherwise, $\mathbb{K}$ would have zero divisors.)

Subfields (I)

$\mathbb{F}_p$ is always a subfield of any finite field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p$. Therefore, $\mathbb{K}$ is a $\mathbb{F}_p$-vector space, and has cardinality $q = p^e$ for some $e \geq 1$.

Then any finite field is a finite **extension** of a prime field.

Similarly, if $\mathbb{K}$ is a finite field and $\mathbb{K}'$ is a subfield, then $\mathbb{K}$ is a $\mathbb{K}'$-vector space. Therefore, if $\mathbb{K}$ has $q = p^e$ elements then $\mathbb{K}'$ has $q' = p^d$ elements for some $d \mid e$.

E.g., a field with 2^{15} elements can only have proper subfields of cardinalities 2, 2^3 and 2^5.

Primitive Elements

Theorem

For any finite field $\mathbb{K}$, the multiplicative group $\mathbb{K}^\times$ is cyclic.

Assume $\mathbb{K}$ has p^e elements. Since $\mathbb{K}^\times$ is an abelian finite group, it is the direct product of r cycles of lengths $l_1, l_2 \ldots, l_r$ fulfilling $l_1 \mid l_2 \mid \cdots \mid l_r$.
But the order of the group is $p^e - 1 = l_1 l_2 \cdots l_r$, and all $x \in \mathbb{K}^\times$ fulfils $x^{l_r} = 1$.
Therefore, the polynomial $X^{l_r} - 1 \in \mathbb{K}[X]$ has at least $p^e - 1$ roots, which is only possible if $l_r \geq p^e - 1$, and then $r = 1$.

Every generator α of $\mathbb{K}^\times$ is called a **primitive** element.

In particular, we can write $\mathbb{K} = \{0, 1 = \alpha^0, \alpha^1, \ldots, \alpha^{p^e-2}\}$

Fröbenius Automorphism (I)

Given a finite field $\mathbb{K}$ of characteristic p, the map $\phi_p : \mathbb{K} \to \mathbb{K}$ defined as $\phi_p(x) = x^p$ is a field automorphism.

$\phi_p(0) = 0$ and ϕ_p is a bijection in $\mathbb{K}^\times$, as $\gcd(p, p^e - 1) = 1$
$\phi_p(xy) = (xy)^p = x^p y^p = \phi_p(x)\phi_p(y)$

$$\phi_p(x+y) = (x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^i y^{p-i} = x^p + y^p = \phi_p(x) + \phi_p(y)$$

since for any prime p, and $0 < i < p$, $\binom{p}{i} \equiv 0 \pmod{p}$.

Moreover, for any polynomial $g \in \mathbb{F}_q[X]$ and $x \in \mathbb{K}$,
$\phi_p(g(x)) = g(\phi_p(x))$ (since elements in $\mathbb{F}_p$ are fixed by ϕ_p).
Thus, if α is a root of g, so do $\alpha^p, \alpha^{p^2}, \ldots$. We call all of them **conjugate** roots.

Fröbenius Automorphism (II)

We can do the same thing with respect to a subfield $\mathbb{K}'$ with cardinality $q = p^d$ of a finite field $\mathbb{K}$ of characteristic p.

$\phi_q : \mathbb{K} \to \mathbb{K}$ defined as $\phi_q(x) = x^q$ is a field automorphism, which fixes the elements in $\mathbb{K}'$. ($\phi_q = \phi_p \circ \cdots \circ \phi_p$)

Moreover, for any polynomial $g \in \mathbb{K}'[X]$ and $x \in \mathbb{K}$,
$\phi_q(g(x)) = g(\phi_q(x))$
Thus, if α is a root of g, so do $\alpha^q, \alpha^{q^2}, \ldots$.

Theorem (Galois)

For any finite field $\mathbb{K}$ and subfield $\mathbb{K}'$, the automorphisms of $\mathbb{K}$ that fix $\mathbb{K}'$ are exactly the powers of ϕ_q, where $q = |\mathbb{K}'|$.

Subfields (II)

Given a finite field $\mathbb{K}$ with p^e elements, all $x \in \mathbb{K}$ satisfies $x^{p^e} = x$, that is, the elements of $\mathbb{K}$ are the roots of $X^{p^e} - X$, and then

$$X^{p^e} - X = \prod_{x \in \mathbb{K}} (X - x) = X \prod_{i=0}^{p^e - 2} (X - \alpha^i)$$

where α is a primitive element of $\mathbb{K}$.

Now, if $\mathbb{K}'$ is a subfield with p^d elements, all $x \in \mathbb{K}'$ also satisfies $x^{p^d} = x$. Then, $\mathbb{K}'$ is the set of roots of $X^{p^d} - X$.

Lemma

$$X^{p^d} - X \mid X^{p^e} - X \quad \Leftrightarrow \quad p^d - 1 \mid p^e - 1 \quad \Leftrightarrow \quad d \mid e$$

Minimal Polynomials

Given a finite field $\mathbb{K}$ with p^e elements, for any nonzero $x \in \mathbb{K}$ we define its **minimal polynomial**, $m_x \in \mathbb{F}_p[X]$ as the monic polynomial of minimal degree such that $m_x(x) = 0$. (It is the monic generator of the ideal of the polynomials vanishing at x.)

m_x is irreducible in $\mathbb{F}_p[X]$ and its roots are the conjugates of x, i.e. $x, x^p, x^{p^2}, \ldots, x^{p^{d-1}}$, where d is the degree of m_x.

Then, $x^{p^d} = x$ and $x^{p^e} = x$, which is only possible if $d \mid e$.
Moreover, x lies in the subfield of q^d elements.

Summarizing,

$$m_x = \prod_{i=0}^{d-1} (X - x^{p^i}) \qquad \text{for some } d \mid e$$

Irreducible Polynomials

Lemma

A polynomial $g \in \mathbb{F}_p[X]$ of degree $e > 0$ is irreducible if and only if $g \mid X^{p^e} - X$, but $g \nmid X^{p^d} - X$ for all nontrivial $d \mid e$.

Corollary

The irreducible factors of $X^{p^e} - X$ in $\mathbb{F}_p[X]$ are exactly the irreducible polynomials in $\mathbb{F}_p[X]$ of degree dividing e

Let $n_e(p)$ be the no. of irred. polynomials of degree e in $\mathbb{F}_p[X]$.

Corollary

$$\sum_{d|e} d n_d(p) = p^e \quad \text{and} \quad 0 \leq \frac{p^e - 2p^{\lfloor e/2 \rfloor}}{e} < n_e(p) \leq \frac{p^e - p}{e}$$

Construction of Finite Fields

The quotient ring $\mathbb{F}_p[X]/g\mathbb{F}_p[X]$ is a field if and only g is an irreducible polynomial in $\mathbb{F}_p[X]$. If g has degree e, then the field has p^e elements.

Theorem

For every prime p and for every $e \geq 1$ there exists a unique (up to isomorphism) finite field with p^e elements.

The finite field with p^e elements, denoted as $GF(p^e)$ or $\mathbb{F}_{p^e}$, can always be constructed as the quotient ring $\mathbb{F}_p[X]/g\mathbb{F}_p[X]$, for any irreducible polynomial $g \in \mathbb{F}_p[X]$ of degree e.

Example: $\mathbb{F}_{2^8}$

$\mathbb{F}_2 = \{0, 1\}$

$g = X^8 + X^4 + X^3 + X^2 + 1$ is irreducible in $\mathbb{F}_2[X]$

$\mathbb{F}_{2^8} = \mathbb{F}_2[X]/g\mathbb{F}_2[X]$

Addition:
$(X^7 + X^4 + X) + (X^6 + X^5 + X^4 + 1) = X^7 + X^6 + X^5 + X + 1$

$10010010 + 01110001 = 11100011$

Multiplication: $(X^7 + X^4 + X) * (X^6 + X^5 + X^4 + 1) =$
$X^7 + X + 1 + (X^5 + X^4 + X^3 + X^2 + 1)g$

$10010010 * 01110001 = 10000011$

Example: $\mathbb{F}_{2^8}$

Primitive element: $\alpha = X$ or 00000010

$|\mathbb{F}_{2^8}^{\times}| = 2^8 - 1 = 255 = 3 * 5 * 17$

$\alpha^{255/17} = X^5 + X^2 + X$ or 00100110

$\alpha^{255/5} = X^3 + X$ or 00001010

$\alpha^{255/3} = X^7 + X^6 + X^4 + X^2 + X$ or 11010110

$\alpha^{255} = 1$ or 00000001

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END OF PART V