Codes and Cryptography

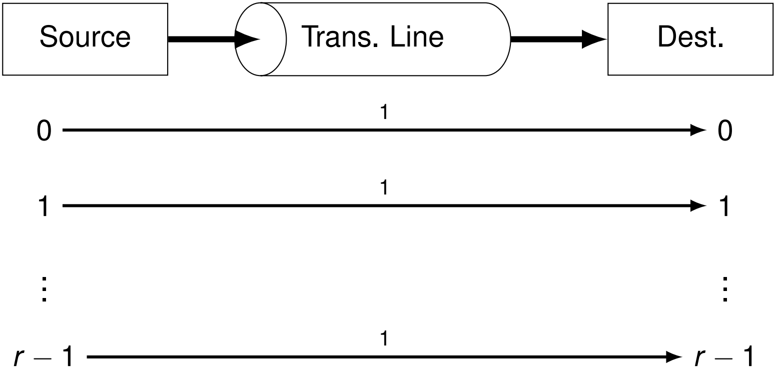
Jorge L. Villar

MAMME, Fall 2015

PART IV

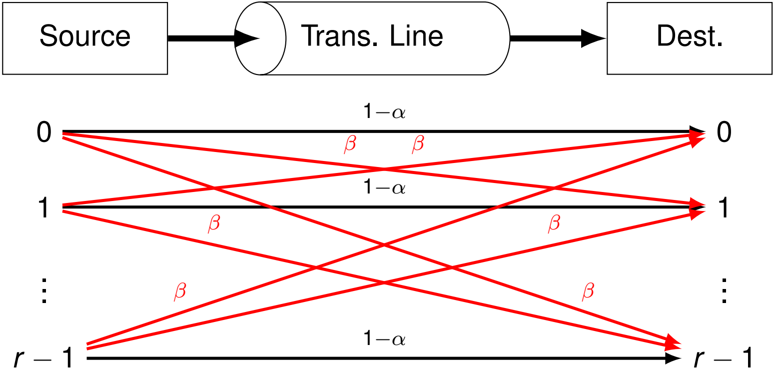
Outline

Transmission Channels



r-ary perfect channel: no transmission errors

Transmission Channels



r-ary symmetric channel $\quad \beta = \frac{\alpha}{r-1}$

Transmission Channels: The Model

Input and output alphabets: $\{0, 1, \ldots, r-1\}$

Transmission probability matrix:

$$T = \begin{pmatrix} t_{0,0} & \cdots & t_{0,r-1} \\ \vdots & \ddots & \vdots \\ t_{r-1,0} & \cdots & t_{r-1,r-1} \end{pmatrix} \text{ where } t_{i,j} = \Pr(\text{receive } j \mid \text{send } i)$$

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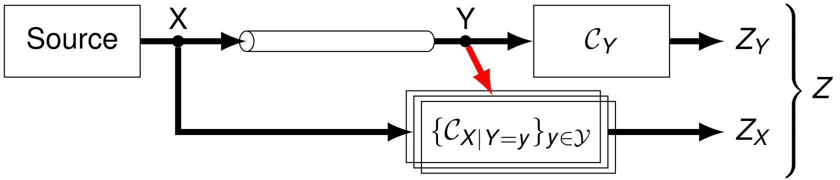
Properties:

- $T \geq 0$
- $T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$
- If $p_j = \Pr(\text{send } j)$ and $q_i = \Pr(\text{receive } i)$, $(q_0 \ \ldots \ q_{r-1}) = (p_0 \ \ldots \ p_{r-1})T$

Channel Capacity (I)



$$L(\mathcal{C}_{X,Y}, \mathcal{S}_{X,Y}) = L(\mathcal{C}_Y, \mathcal{S}_Y) + \sum_{y\in\mathcal{Y}} \Pr(Y = y) L(\mathcal{C}_{X|Y=y}, \mathcal{S}_{X|Y=y})$$

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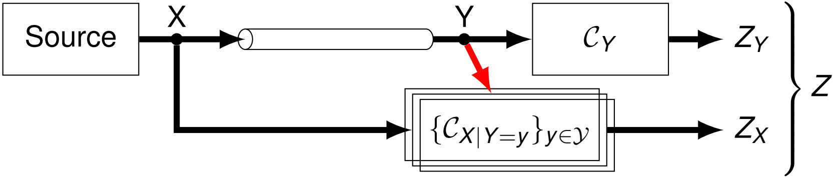

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If all (binary) codes are optimal then

$H(Y) \le L(\mathcal{C}_Y,\mathcal{S}_Y) < H(Y)+1$

$H(X \mid Y=y) \le L(\mathcal{C}_{X|Y=y},\mathcal{S}_{X|Y=y}) < H(X \mid Y=y)+1$

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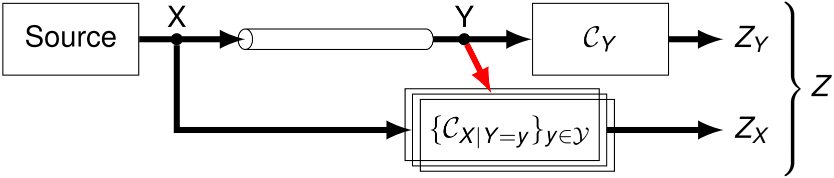

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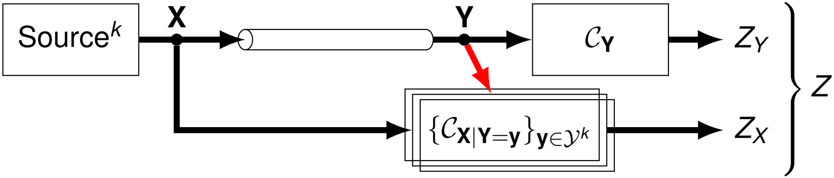
Thus,

$H(X,Y) = H(Y) + H(X \mid Y) \leq L(\mathcal{C}_{X,Y},\mathcal{S}_{X,Y}) < H(X,Y) + 2$

Channel Capacity (II)

With extended sources,



$$L(\mathcal{C}_{\mathbf{X},\mathbf{Y}}, \mathcal{S}^k_{X,Y}) = L(\mathcal{C}_{\mathbf{Y}}, \mathcal{S}^k_{Y}) + \sum_{\mathbf{y}\in\mathcal{Y}^k} \Pr(\mathbf{Y}=\mathbf{y}) L(\mathcal{C}_{\mathbf{X}|\mathbf{Y}=\mathbf{y}}, \mathcal{S}^k_{X|\mathbf{Y}=\mathbf{y}})$$

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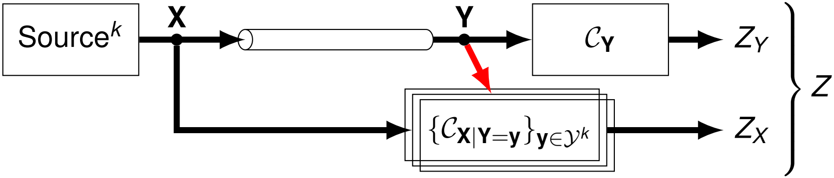

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Now,

$$H(\mathbf{X},\mathbf{Y}) = H(\mathbf{Y}) + H(\mathbf{X}\mid\mathbf{Y}) = kH(Y) + kH(X\mid Y) =$$
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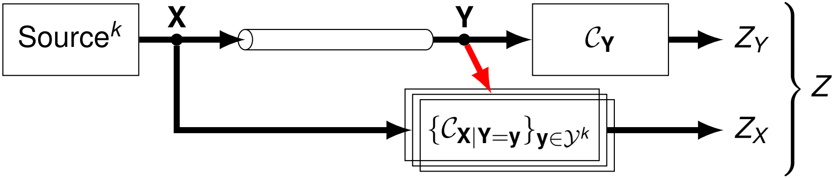

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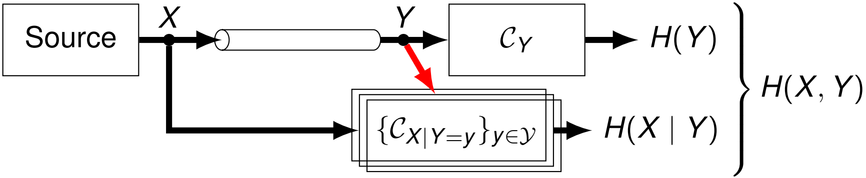
$kH(X,Y) \leq L(\mathcal{C}_{\mathbf{X},\mathbf{Y}}, \mathcal{S}^k_{X,Y}) < H(\mathbf{X},\mathbf{Y}) + 2 = kH(X,Y) + 2$

Therefore, asymptotically $H(X,Y) = \dfrac{1}{k} L(\mathcal{C}_{\mathbf{X},\mathbf{Y}}, \mathcal{S}^k_{X,Y})$

Channel Capacity (III)

Regarding entropies,



The information about X conveyed by Y is
$H(X) - H(X \mid Y) = H(X) + H(Y) - H(X, Y) = I(X; Y)$

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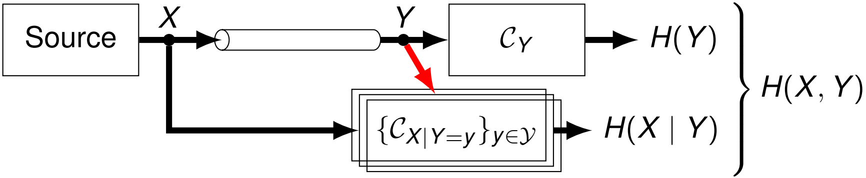

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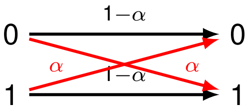
Definition (Channel Capacity)

$$\Lambda = \max_{\mathcal{S}_X}(I(X; Y))$$

where the maximum is taken over all possible sources $\mathcal{S}_X$

Capacity of the Binary Symmetric Channel



For $\Pr(X = 1) = p$, $H(X) = h_p = -p\log_2 p - (1-p)\log_2(1-p)$

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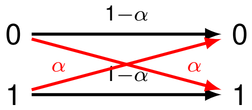

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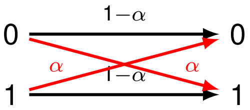

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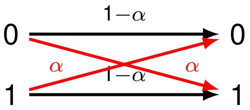

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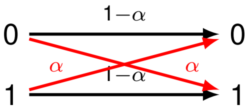

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Conditioned to X, $H(Y\mid X=0)=H(Y\mid X=1)=h_\alpha$.

$I(X;Y)=H(Y)-H(Y\mid X)=h_{p(1-\alpha)+(1-p)\alpha}-h_\alpha$

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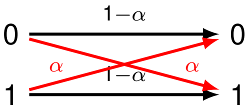

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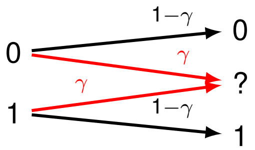
The capacity of the BSC is $\Lambda_{BSC} = 1 - h_\alpha$

Capacity of the Binary Channel With Erasures



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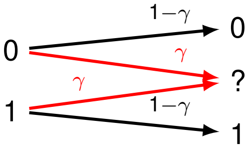

For $\Pr(X=1)=p$, $H(X)=h_p=-p\log_2 p-(1-p)\log_2(1-p)$

$\Pr(Y=0)=p(1-\gamma)$, $\Pr(Y=1)=(1-p)(1-\gamma)$,
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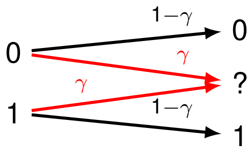

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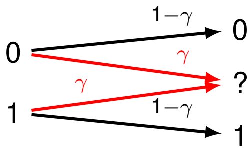

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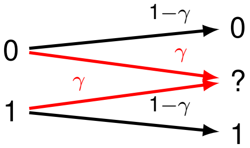

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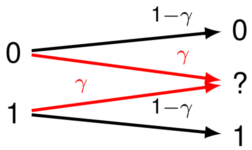

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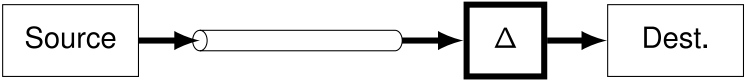
The capacity of the BSEC is $\Lambda_{BSEC}=1-\gamma$

Outline

Decision Rules



Decision rule Δ: Guess the symbol sent from the received symbol

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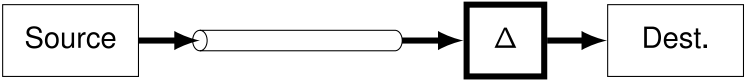

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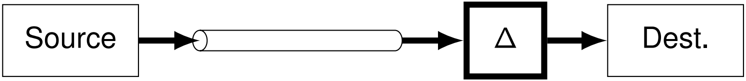

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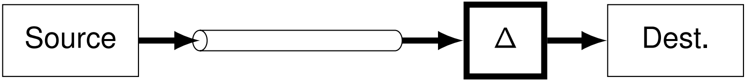

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Both are equivalent when the error probabilities are small and the source is balanced enough

The BSC Case

Transmission matrix: $T = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix}$

Max. Likelihood:

$Y = 0 \quad \longrightarrow \quad \text{if } \ 1-\alpha \geq \alpha \ \text{ then } \ \widehat{X} = 0$

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$p_{corr} = \max(\alpha, p, 1-p, 1-\alpha)$

Both differ only when the channel is useless!

Improving Channel Reliability (I)

GOAL: Make p_{corr} as close to 1 as possible maintaining an information transmission rate close to the theoretical channel capacity

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E.g., the information rate of the uniform binary source is $H(X) = 1$, but the capacity of the BSC is $\Lambda_{BSC} = 1 - h_\alpha$. Therefore, we need to expand X to a length of at least $\frac{1}{\Lambda_{BSC}}$ to make $p_{corr} \to 1$.

Improving Channel Reliability (II)

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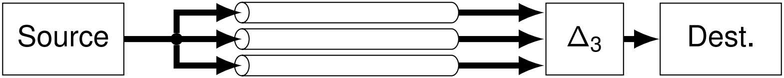
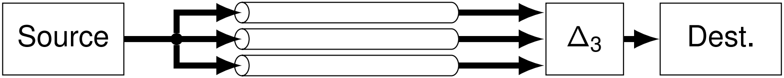
Channel replication = Repetition code

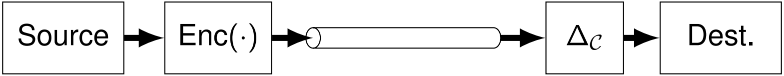


Improving Channel Reliability (II)

Channel replication = Repetition code



"Encode then Transmit"



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Given two words $x, y \in \{0, \ldots, r-1\}^n$, the Hamming distance $d(x, y)$ is the number positions in which they differ.

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$$p_{corr} = \Pr(\widehat{X} = X) = \Pr(d(Y, X^n) \leq \ell) = \sum_{i=0}^{\ell} \binom{n}{i} \alpha^i (1-\alpha)^{n-i}$$

For $\alpha = 0.1$, $\Lambda_{BSC} = 0.531$ but for $p_{corr} \geq 0.999$ we need $n = 9$

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$$\Pr(\widehat{X} = \mathbf{x} \mid X = \mathbf{x}) = \Pr(Y \in \text{neigh}(E_{\mathcal{C}_k}(\mathbf{x})) \mid X = \mathbf{x}) \ \text{ where}$$

$$\text{neigh}(\mathbf{w}) = \{\mathbf{y} \in \{0,1\}^n \ : \ \forall \mathbf{z} \in \mathcal{C}_k \setminus \{\mathbf{w}\},\ d(\mathbf{y},\mathbf{z}) > d(\mathbf{y},\mathbf{w})\}$$

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$$\Pr(\widehat{X} = \mathbf{x} \mid X = \mathbf{x}) = \sum_{\mathbf{y} \in neigh(E_{\mathcal{C}_k}(\mathbf{x}))} \alpha^{d(\mathbf{y},E_{\mathcal{C}_k}(\mathbf{x}))} (1-\alpha)^{n-d(\mathbf{y},E_{\mathcal{C}_k}(\mathbf{x}))}$$

Outline

The Rate of a Constant-Length Code

Definition (Code Rate)

The rate of a constant-length r-ary code $\mathcal{C} \subset \{0, \ldots, r-1\}^n$ is

$$R(\mathcal{C}) = \frac{\log_r(|\mathcal{C}|)}{n}$$

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For a constant-length r-ary code $\mathcal{C}_k \subset \{0, \dots, r-1\}^n$ for an r-ary extended source $\mathcal{S}^k$, $|\mathcal{C}_k| = r^k$ and $R(\mathcal{C}_k) = \frac{k}{n}$

GOAL(restated)**:** Given an r-ary channel with capacity Λ, find $\mathcal{C}$ with $p_{corr} \approx 1$ and $R(\mathcal{C}) \log_2 r \approx \Lambda$.

Shannon's Theorem

Theorem (Shannon)

Given an r-ary channel (stateless and source independent) with capacity Λ, for any positive $\varepsilon, \delta \in \mathbb{R}^+$ there exists $n_0 \in \mathbb{Z}^+$ such that for all $n > n_0$ there exists an r-ary code $\mathcal{C}$ of constant-length n, and a decision rule for it such that

$$\Lambda - \epsilon < R(\mathcal{C}) \log_2 r < \Lambda \qquad p_{corr} > 1 - \delta$$

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The proof builds on the idea that random codes behave well with high probability, but it is not constructive.

Sketch of the proof for the BSC (I)

Consider a random binary code $\mathcal{C}$ of constant-length n and rate R (i.e., $|\mathcal{C}| = 2^{nR}$). Now a random codeword $X = (X_1, \dots, X_n)$ is sent through a BSC with error probability $\alpha < \frac{1}{2}$. The word at reception is $Y = (Y_1, \dots, Y_n)$. We use the *nearest-codeword* decoding rule.

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The expected value of p_{corr} is

$$\mathrm{E}(p_{corr}) = \frac{1}{2^n} \sum_{\mathbf{x},\mathbf{y} \in \{0,1\}^n} t_{\mathbf{x},\mathbf{y}} p_{corr}(\mathbf{x}, \mathbf{y})$$

where $t_{\mathbf{x},\mathbf{y}} = \Pr(Y = \mathbf{y} \mid X = \mathbf{x}) = (1 - \alpha)^{n-d(\mathbf{x},\mathbf{y})} \alpha^{d(\mathbf{x},\mathbf{y})}$ and $p_{corr}(\mathbf{x}, \mathbf{y})$ is the probability that $\forall \mathbf{z} \in \mathcal{C} \setminus \{\mathbf{x}\},\ d(\mathbf{z}, \mathbf{y}) > d(\mathbf{x}, \mathbf{y})$.

Sketch of the proof for the BSC (II)

As $p_{corr}(\mathbf{x},\mathbf{y})$ only depends on $d(\mathbf{x},\mathbf{y})$,

$$\mathrm{E}(p_{corr}) = \sum_{d=0}^{n} \binom{n}{d} \alpha^d (1-\alpha)^{n-d} p_{corr}(d)$$

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$$p_{corr}(d) = \left(1 - \frac{1+n+\ldots+\binom{n}{d}}{2^n}\right)^{|\mathcal{C}|-1}$$

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Recall that $|\mathcal{C}| = 2^{nR}$. Then, for large enough n and $d \leq \frac{n}{2}$

$$p_{corr}(d) \approx e^{-2^{n(R-1)}(1+n+\ldots+\binom{n}{d})}$$

Sketch of the proof for the BSC (III)

Lemma

For any $d \leq \frac{n}{2}$, $\quad 1 + n + \ldots + \binom{n}{d} \leq 2^{nh_{d/n}}$ *where*
$h_\alpha = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$

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that is, the probability that a binomial(n, α) random variable is less than γn, which tends to 1 whenever $\gamma > \alpha$.

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that is, the probability that a binomial(n, α) random variable is less than γn, which tends to 1 whenever $\gamma > \alpha$. Equivalently, $R = 1 - h_\gamma < 1 - h_\alpha = \Lambda_{BSC}$

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Theorem (Fano Bound)

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Then, $p_{corr} = \sum_{(i,j)\in\Delta} p_i t_{i,j}$ and $H(X \mid Y) = -\sum_{(i,j)} p_i t_{i,j} \log_2 \frac{p_i t_{i,j}}{q_j}$

or $H(X \mid Y) = \sum_{(i,j)\in\Delta} p_i t_{i,j} \log_2 \frac{q_j}{p_i t_{i,j}} + \sum_{(i,j)\notin\Delta} p_i t_{i,j} \log_2 \frac{q_j}{p_i t_{i,j}}$

Fano Bound (II)

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$$Q \ln 2 \le p_{corr} \sum_{(i,j)\in\Delta} q_j - \sum_{(i,j)\in\Delta} p_i t_{i,j} + \frac{(1 - p_{corr})}{(r - 1)} \sum_{(i,j)\notin\Delta} q_j - \sum_{(i,j)\notin\Delta} p_i t_{i,j}$$

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But $\sum_{(i,j)\in\Delta} q_j = 1$ and $\sum_{(i,j)\notin\Delta} q_j = r-1$ both imply $Q \ln 2 \leq 0$

Converse of Shannon's Theorem

Theorem

Given an r-ary channel (stateless and source independent) with capacity Λ *and a uniform source, for any* $\Lambda_1 > \Lambda_2 > \Lambda$ *there exists no sequence of codes* $\{\mathcal{C}_n\}_{n \geq n_0}$ *of constant length n and rates* R_n *such that* $\Lambda_1 > R_n \log_2 r > \Lambda_2$ *and* $p_{corr} \to 1$ *as* $n \to \infty$.

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For X uniformly distributed in $\mathcal{C}_n$, using Fano bound

$$nR_n \log_2 r = H(X) = H(X \mid Y) + I(X;Y) \leq$$
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Thus, $p_{corr} \leq \dfrac{1 + n\Lambda}{nR_n \log_2 r} \leq \dfrac{1 + n\Lambda}{n\Lambda_2}$

Finally, for $n > \dfrac{2}{\Lambda_2 - \Lambda}$, p_{corr} cannot exceed $1 - \dfrac{\Lambda_2 - \Lambda}{2\Lambda_2}$.

Remarks

A transmission channel has a maximum transmission rate, the channel capacity Λ.

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There are also some combinatorial bounds limiting the existence of codes with practical lengths.

The goal is now build not-too-large efficiently decodable codes with good p_{corr} and rate.

Codes and Cryptography

Jorge L. Villar

MAMME, Fall 2015

END OF PART IV

Proof of the Lemma

For any $\alpha \leq \frac{1}{2}$ and $d \leq n$

$$1 = \sum_{i=0}^{n} \binom{n}{i} \alpha^i (1-\alpha)^{n-i} = (1-\alpha)^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{\alpha}{1-\alpha}\right)^i \geq$$

$$\geq (1-\alpha)^n \sum_{i=0}^{d} \binom{n}{i} \left(\frac{\alpha}{1-\alpha}\right)^d = \alpha^d (1-\alpha)^{n-d} \sum_{i=0}^{d} \binom{n}{i}$$

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Thus, $\displaystyle\sum_{i=0}^{d} \binom{n}{i} \leq \alpha^{-d} (1-\alpha)^{-(n-d)}$

Now, if $d \leq \frac{n}{2}$ we can set $\alpha = \frac{d}{n}$, that is $d = n\alpha$, and then

$$\sum_{i=0}^{d} \binom{n}{i} \leq \left(\alpha^\alpha (1-\alpha)^{1-\alpha}\right)^{-n} = 2^{nh_\alpha}$$

go back...