Codes and Cryptography

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PART III

Outline

Sources

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- Probability distribution $(p_1, \ldots, p_n)$ where p_i is the probability of symbol α_i

We simply say $\mathcal{S} = (p_1, \ldots, p_n)$

Optimal Codes (I)

Optimal code: an r-ary uniquely-decodable code $\mathcal{C}$ that minimizes the average codeword length

$$L(\mathcal{C}, \mathcal{S}) = \sum_{i=1}^{n} l(w_i) p_i$$

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There always exists a prefix-free optimal code

Optimal Codes (II)

We solve the continuous optimization problem:

Minimize $L = \sum_{i=1}^{n} l_i p_i$ for $l_1, \ldots, l_n$ **subject to** $\sum_{i=1}^{n} r^{-l_i} \leq 1$

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As L is a linear function, its minima are on the boundary. Now, using Lagrange multipliers

$$\nabla L = (p_1, \ldots, p_n) = \lambda \nabla \left(\sum_{i=1}^{n} r^{-l_i} - 1 \right) = -\lambda \ln r (r^{-l_1}, \ldots, r^{-l_n})$$

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Therefore $l_i = -\log_r p_i$ and $L_{opt} = -\sum_{i=1}^{n} p_i \log_r p_i$

Shannon's Lower Bound

We have find the following lower bound

Theorem (Shannon)

For any stateless stationary source $\mathcal{S} = (p_1, \ldots, p_n)$ *and any* r*-ary uniquely-decodable code* $\mathcal{C}$

$$L(\mathcal{C}, \mathcal{S}) \geq L_r(\mathcal{S}) = -\sum_{i=1}^{n} p_i \log_r p_i$$

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However, this lower bound is achievable if and only if all $l_i = -\log_r p_i$ are integers (i.e., p_i are negative powers of r).

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Now $L_r(\mathcal{S}) = -n\frac{1}{n}\log_r\left(\frac{1}{n}\right) = \log_r n = d$ is achievable by taking $l_i = -\log_r\left(\frac{1}{n}\right) = d$

The optimal code is the constant length code $\mathcal{C} = \{0, \ldots, r-1\}^d$

Shannon-Fano Codes

A good approximation for the optimal (continuous) case is taking $l_i = \lceil -\log_r p_i \rceil$.

There always exists a prefix-free code, the Shannon-Fano code, for this codeword length sequence, since

$$\sum_{i=1}^{n} r^{-l_i} = \sum_{i=1}^{n} r^{-\lceil -\log_r p_i \rceil} \leq \sum_{i=1}^{n} r^{\log_r p_i} = \sum_{i=1}^{n} p_i = 1$$

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On the other hand, the code is near-optimal

$$L(\mathcal{C}, \mathcal{S}) = \sum_{i=1}^{n} p_i \lceil -\log_r p_i \rceil < \sum_{i=1}^{n} p_i(-\log_r p_i + 1) = L_r(\mathcal{S}) + 1$$

Source Coalescence

Consider two sources $\mathcal{S}_1 = (p_1, \ldots, p_n)$, $\mathcal{S}_2 = (q_1, \ldots, q_m)$. We can define a new source $\mathcal{S}_1 \times \mathcal{S}_2 = (p_1 q_1, p_1 q_2, \ldots, p_n q_m)$ for the cartesian product of the two source alphabets.

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However, $L_r(\mathcal{S}_1 \times \mathcal{S}_2) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j \log_r(p_i q_j) =$

$$-\sum_{i=1}^{n} p_i \log_r p_i - \sum_{j=1}^{m} q_j \log_r q_j = L_r(\mathcal{S}_1) + L_r(\mathcal{S}_2)$$

Source Extensions (I)

Given a source $\mathcal{S} = (p_1, \ldots, p_n)$ we can define its k-th extension $\mathcal{S}^k = \underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{k \text{ times}}$.

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Now, $L_r(\mathcal{S}^k) = kL_r(\mathcal{S})$ and the Shannon-Fano code $\mathcal{C}_k$ for $\mathcal{S}^k$ satisfies $L_r(\mathcal{S}^k) \leq L(\mathcal{C}_k, \mathcal{S}^k) < L_r(\mathcal{S}^k) + 1$

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But the correct performance measure for k-tuples is

$$L_r(\mathcal{S}) \leq \frac{L(\mathcal{C}_k, \mathcal{S}^k)}{k} < L_r(\mathcal{S}) + \frac{1}{k}$$

In the context of source extensions, **Shannon's lower bound is asymptotically achieved** by Shannon-Fano codes.

Source Extensions (II)

Theorem (Noiseless Coding Theorem [Shannon, 1948])

For any stateless stationary source $\mathcal{S}$ and any family $\{\mathcal{C}_k\}_{k\in\mathbb{Z}^+}$ of r-ary optimal uniquely-decodable codes for $\{\mathcal{S}^k\}_{k\in\mathbb{Z}^+}$

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The proof simply considers the fact that any optimal code performs better than any other code, in particular, better that Shannon-Fano codes

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Construction of Optimal Codes (I)

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Given $\mathcal{S} = (p_1, \ldots, p_n)$ we look for an optimal r-ary prefix-free code $\mathcal{C}$ with some codeword lengths $l_1, \ldots, l_n$. Let's assume that $p_1 \geq \cdots \geq p_n$).

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Lemma (1)

In an optimal code, $p_i > p_j \Rightarrow l_i \leq l_j$.

Otherwise, swapping i-th and j-th codewords in $\mathcal{C}$ would result in a better code.

Construction of Optimal Codes (II)

Let $T(\mathcal{C})$ be the subtree of a complete r-ary tree formed by the paths from the root node to the nodes of the codewords in $\mathcal{C}$.

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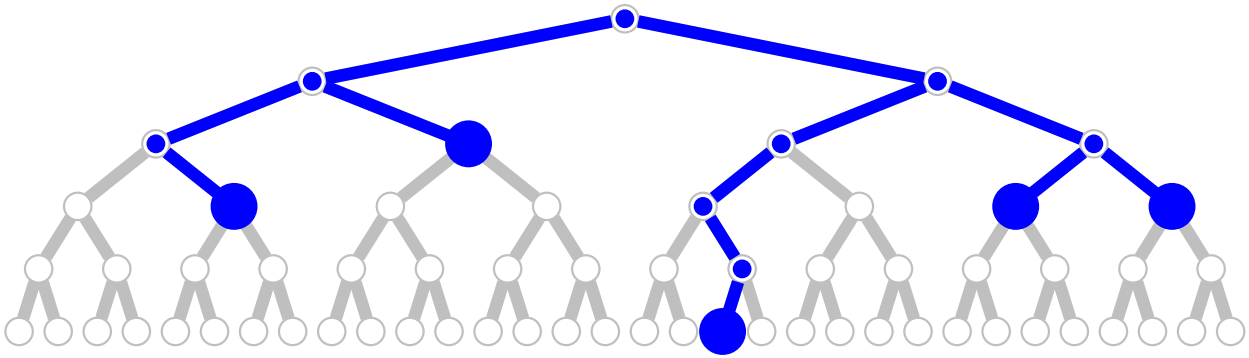
For example, $\mathcal{C} = \{001, 01, 10010, 110, 111\}$



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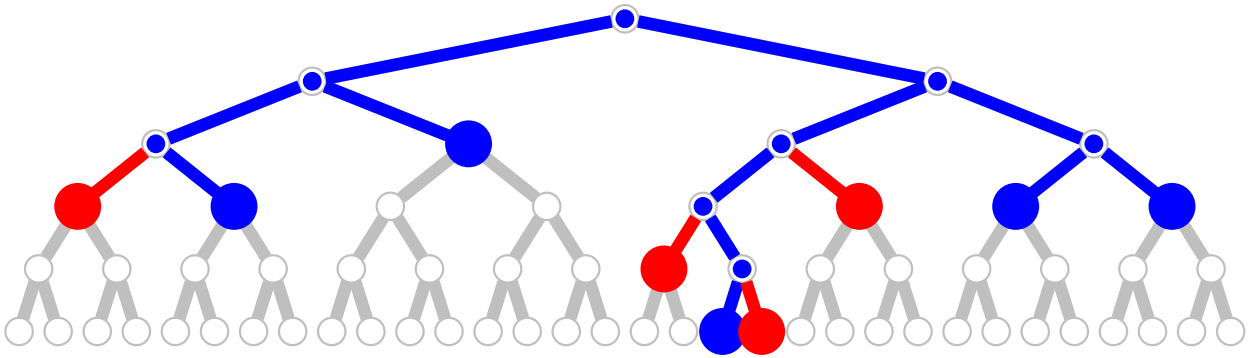

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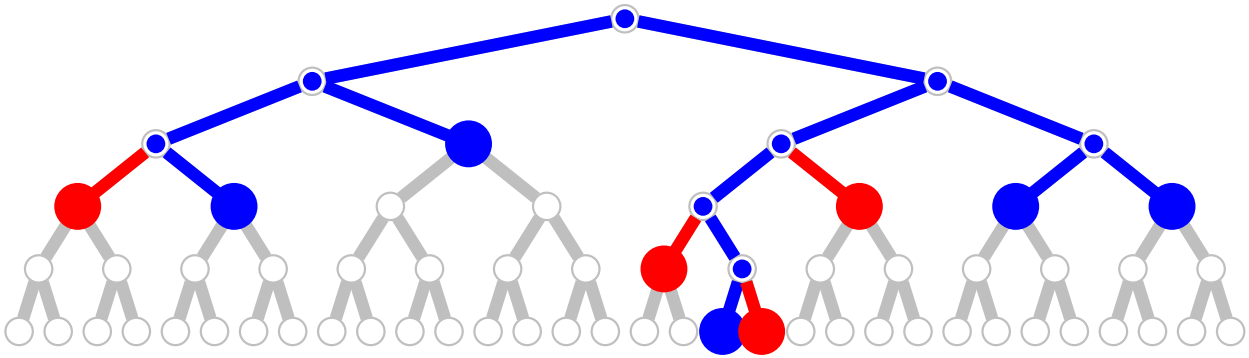

We call **available** the nodes in the complement of $T(\mathcal{C})$ with siblings in $T(\mathcal{C})$. After adding a new codeword corresponding to an available node, $\mathcal{C}$ remains prefix-free!

Construction of Optimal Codes (III)

Lemma (2)

If $\mathcal{C}$ is optimal, then available nodes of $T(\mathcal{C})$ can only occur in the lowest level.

Otherwise, we can change one codeword in the lowest level by the codeword associated to the available node, then improving the code.

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Lemma (3)

There exists an optimal code $\mathcal{C}$ with $(-n+1) \bmod (r-1)$ available nodes, and all of them are siblings and they are in the lowest level.

In the binary case, $r = 2$, there are no available nodes.

Construction of Optimal Codes (IV)

Example: a ternary code

$\mathcal{C} = \{2, 00, 02, 10, 12, 010, 012, 110, 111\}$

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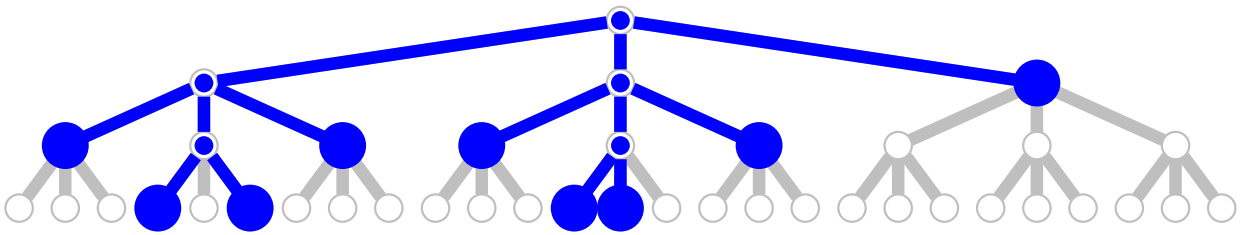

All available nodes are at the lowest level

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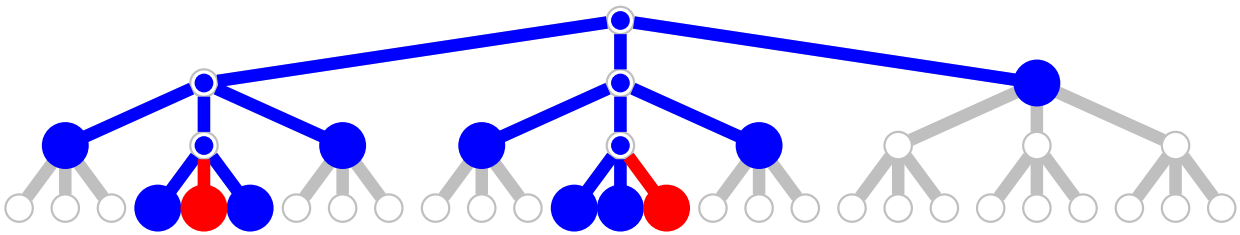

Rearrange lowest level nodes to make all available nodes siblings

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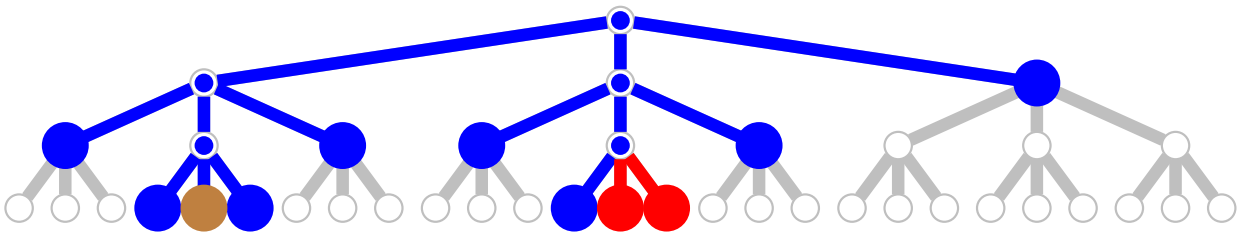

A single child can always be replaced by its parent

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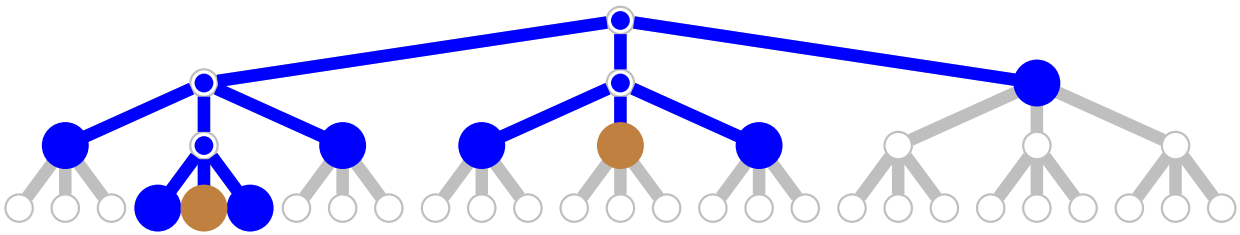

Now the code can be optimal!

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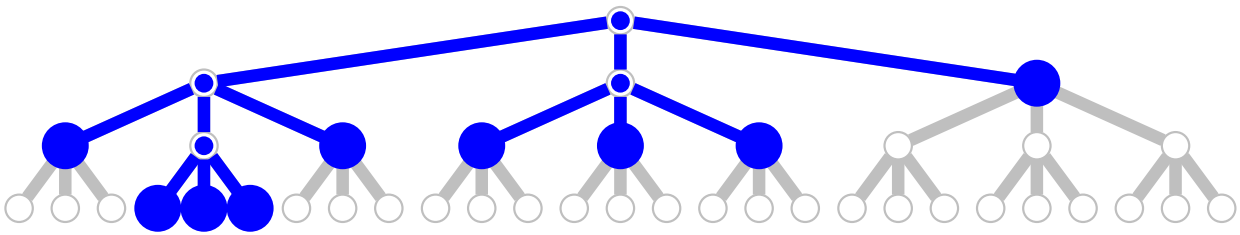

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Still the lowest level nodes can be rearranged to make the set of $r - (-n+1) \bmod (r-1)$ siblings have the lowest probabilities.

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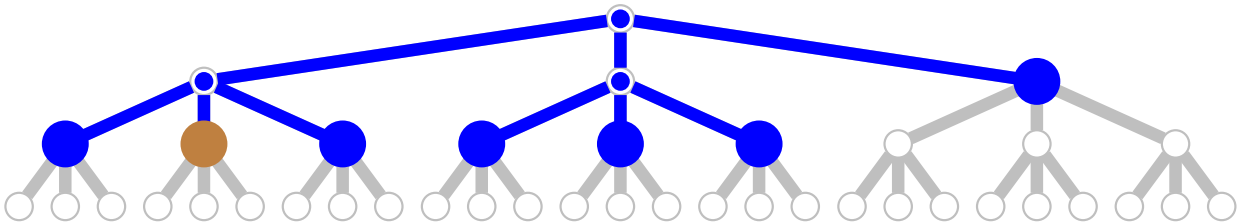

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If they are replaced by their parent, the code should remain optimal!

Huffman Codes: Algorithm

A recursive construction algorithm:

Input: A sequence of probabilities $p_1 \geq \cdots \geq p_n$
Output: An optimal r-ary code $\{w_1, \ldots, w_n\}$.

- If $n \leq r$, then output $\mathcal{C} = \{0, \ldots, n-1\}$. Otherwise, ...
- Compute $m = r - ((-n+1) \bmod (r-1))$.
- Replace $(p_{n-m+1}, \ldots, p_n)$ by $p'_{n-m+1} = p_{n-m+1} + \ldots + p_n$.
- Sort the new sequence and compute an optimal code $\mathcal{C}'$ for it.
- Undo the above sorting procedure and replace the last codeword w by the m codewords $w0, \ldots, w(m-1)$.

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Remark: $m = r$ except perhaps in the first step for $r \geq 3$

Huffman Codes: Example (I)

Example: $r = 3,\ n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

Huffman Codes: Example (I)

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$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

$m = 2$, Sort:

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

$m = 3$, Sort:

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228, 0.0130})$

$m = 3$, Sort:

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769, 0.0520, 0.0510})$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769}, 0.0520, 0.0510)$

$m = 3$, Sort:

[4] $(0.4355, 0.1821, \underbrace{0.1799}, 0.1161, 0.0864)$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228, 0.0130})$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769, 0.0520, 0.0510})$

[4] $(0.4355, 0.1821, \underbrace{0.1799, 0.1161, 0.0864})$

$n = 3 \leq r$, Sort: (not strictly necessary here)

[5] $(0.4355, 0.3824, 0.1821)$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228, 0.0130})$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769, 0.0520, 0.0510})$

[4] $(0.4355, 0.1821, \underbrace{0.1799, 0.1161, 0.0864})$

[5] $(0.4355, 0.3824, 0.1821)$

Assign codewords:

[5] $(0, 1, 2)$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769}, 0.0520, 0.0510)$

[4] $(0.4355, 0.1821, \underbrace{0.1799}, 0.1161, 0.0864)$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

Unsort and expand codewords:

[4] $(0, 2, \overbrace{10}, 11, 12)$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228, 0.0130})$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769, 0.0520, 0.0510})$

[4] $(0.4355, 0.1821, \underbrace{0.1799, 0.1161, 0.0864})$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

[4] $(0, 2, \overbrace{10, 11, 12})$

Unsort and expand codewords:

[3] $(0, 2, 11, 12, \overbrace{100, 101, 102})$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769}, 0.0520, 0.0510)$

[4] $(0.4355, 0.1821, \underbrace{0.1799}, 0.1161, 0.0864)$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

[4] $(0, 2, \overbrace{10, 11, 12})$

[3] $(0, 2, 11, 12, \overbrace{100, 101, 102})$

Unsort and expand codewords:

[2] $(0, 2, 11, 12, 101, 102, \overbrace{1000, 1001, 1002})$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228, 0.0130})$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769, 0.0520, 0.0510})$

[4] $(0.4355, 0.1821, \underbrace{0.1799, 0.1161, 0.0864})$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

[4] $(0, 2, \overbrace{10, 11, 12})$

[3] $(0, 2, 11, 12, \overbrace{100, 101, 102})$

[2] $(0, 2, 11, 12, 101, 102, \overbrace{1000, 1001, 1002})$

Unsort and expand codewords:

[1] $(0, 2, 11, 12, 101, 102, 1000, 1002, \overbrace{10010, 10011})$

Huffman Codes: Example (I)

Example: $r = 3$, $n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769}, 0.0520, 0.0510)$

[4] $(0.4355, 0.1821, \underbrace{0.1799}, 0.1161, 0.0864)$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

[4] $(0, 2, \overbrace{10, 11, 12})$

[3] $(0, 2, 11, 12, \overbrace{100, 101, 102})$

[2] $(0, 2, 11, 12, 101, 102, \overbrace{1000, 1001, 1002})$

[1] $(0, 2, 11, 12, 101, 102, 1000, 1002, \overbrace{10010, 10011})$

Finally, unsort:

$\mathcal{C} = (2, 11, 102, 1002, 10010, 0, 12, 1000, 10011, 101)$

Huffman Codes: Example (I)

Example: $r = 3,\ n = 10$

$\mathcal{S} = (0.1821, 0.1161, 0.0510, 0.0130, 0.0124, 0.4355, 0.0864, 0.0411, 0.0104, 0.0520)$

[1] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, 0.0411, 0.0130, \underbrace{0.0124, 0.0104})$

[2] $(0.4355, 0.1821, 0.1161, 0.0864, 0.0520, 0.0510, \underbrace{0.0411, 0.0228}, 0.0130)$

[3] $(0.4355, 0.1821, 0.1161, 0.0864, \underbrace{0.0769}, 0.0520, 0.0510)$

[4] $(0.4355, 0.1821, \underbrace{0.1799, 0.1161, 0.0864})$

[5] $(0.4355, 0.3824, 0.1821)$

[5] $(0, 1, 2)$

[4] $(0, 2, \overbrace{10, 11, 12})$

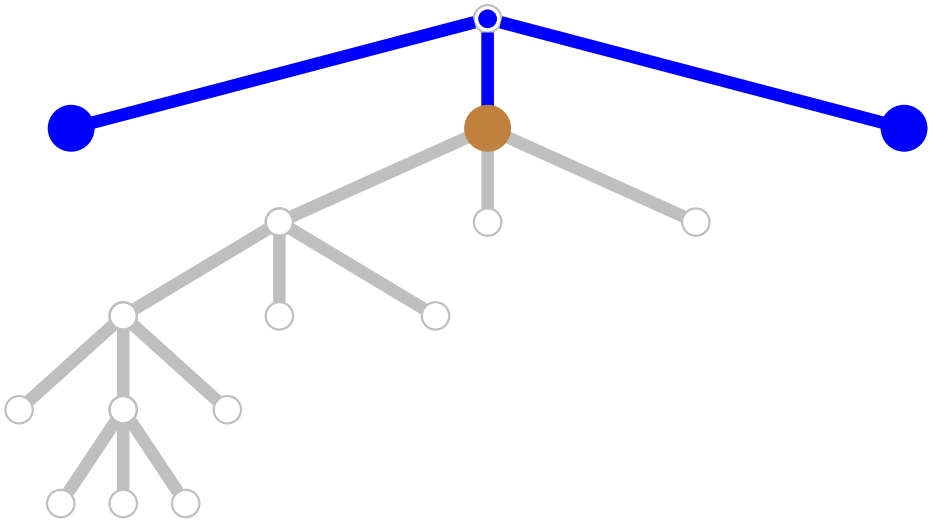
[3] $(0, 2, 11, 12, \overbrace{100, 101, 102})$

[2] $(0, 2, 11, 12, 101, 102, \overbrace{1000, 1001, 1002})$

[1] $(0, 2, 11, 12, 101, 102, 1000, 1002, \overbrace{10010, 10011})$

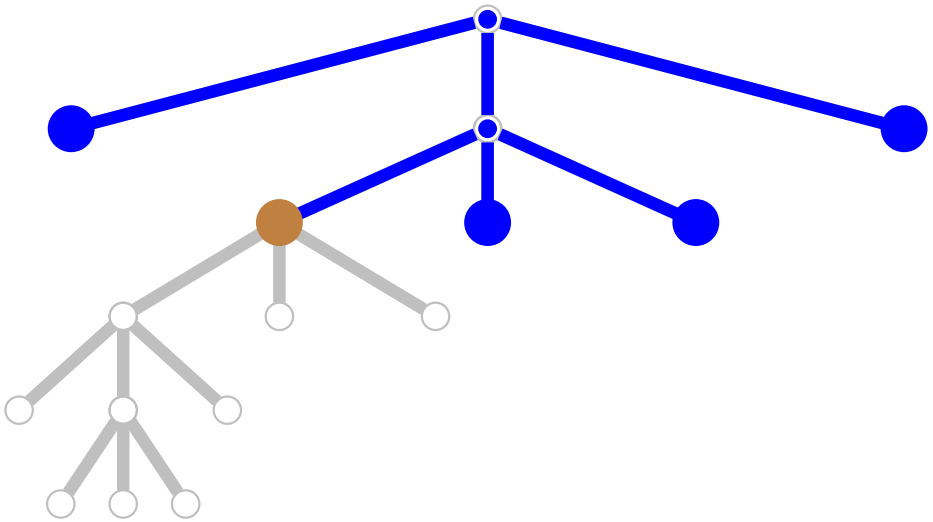
$\mathcal{C} = (2, 11, 102, 1002, 10010, 0, 12, 1000, 10011, 101)$

Huffman Codes: Example (II)



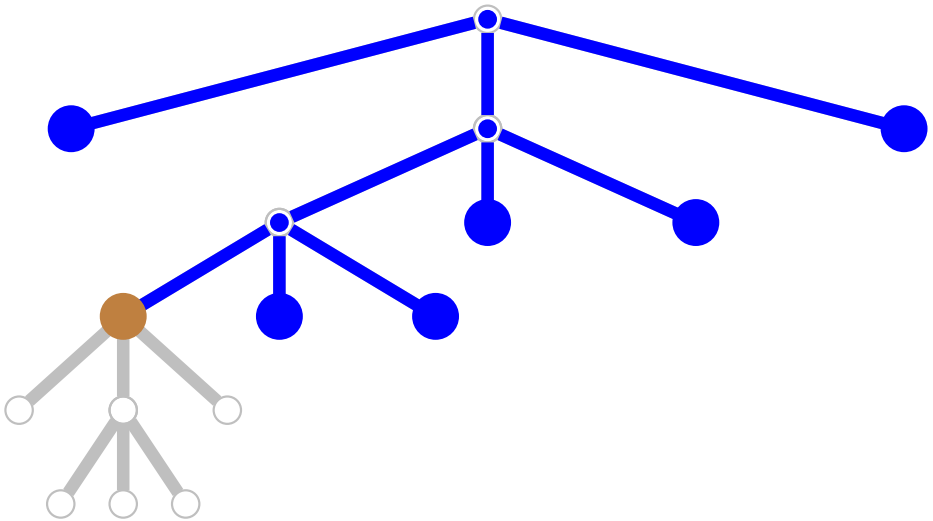
$\mathcal{C}_5 = \{0, 1, 2\}$

Huffman Codes: Example (II)



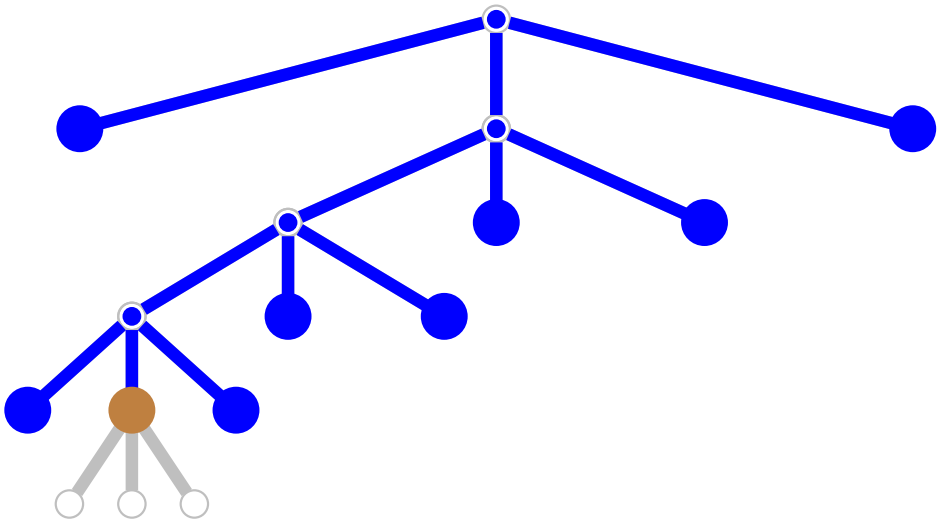
$$\mathcal{C}_4 = \{0, 2, \overbrace{10, 11, 12}\}$$

Huffman Codes: Example (II)



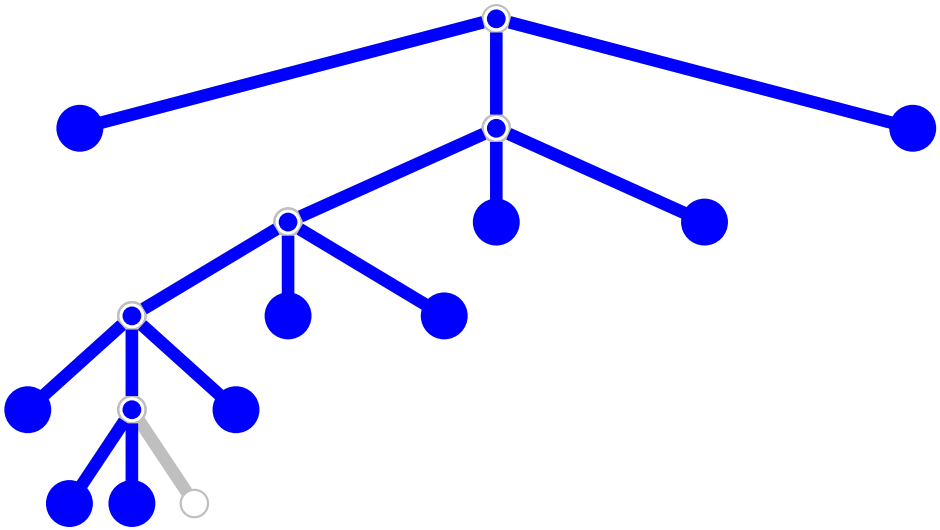
$$\mathcal{C}_3 = \{0, 2, 11, 12, \overbrace{100, 101, 102}\}$$

Huffman Codes: Example (II)



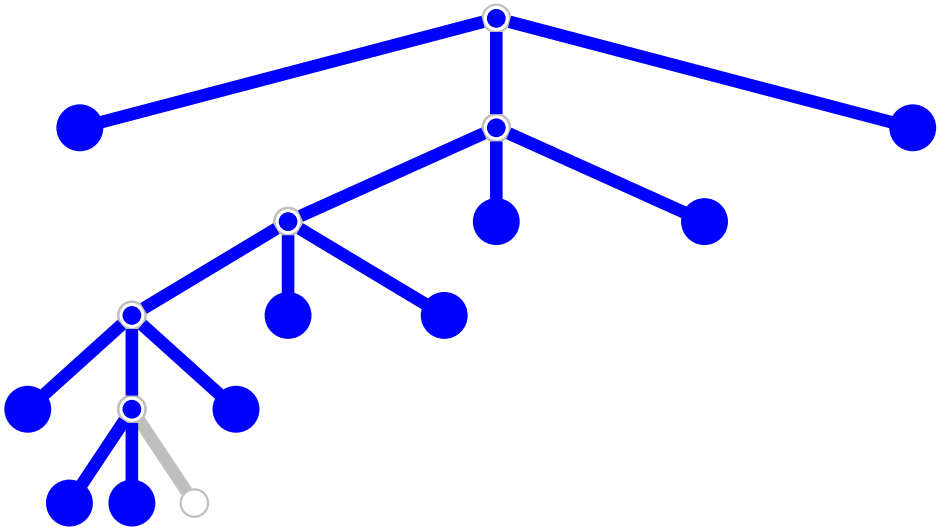
$$\mathcal{C}_2 = \{0, 2, 11, 12, 101, 102, \overbrace{1000, 1001, 1002}\}$$

Huffman Codes: Example (II)



$$\mathcal{C}_1 = \{0, 2, 11, 12, 101, 102, 1000, 1002, \overbrace{10010, 10011}\}$$

Huffman Codes: Example (II)

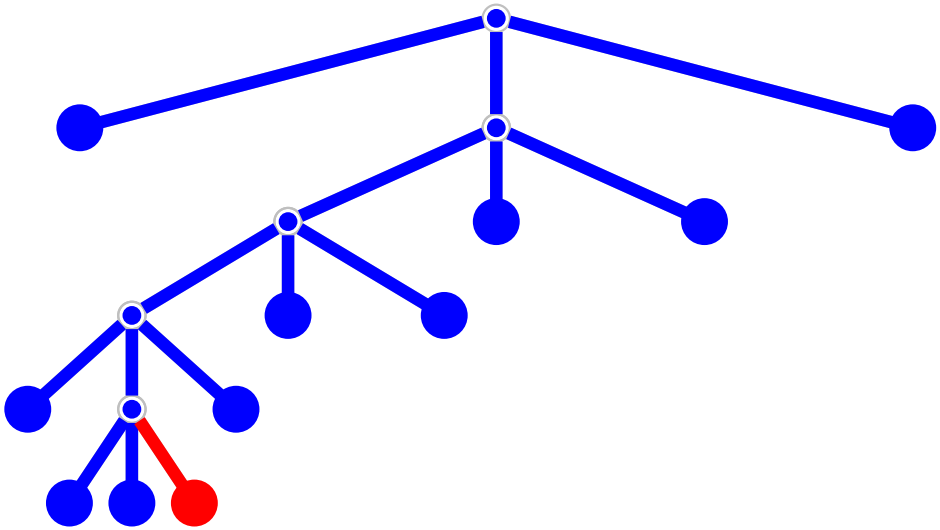


$\mathcal{C} = \{2, 11, 102, 1002, 10010, 0, 12, 1000, 10011, 101\}$

$\mathcal{C}_{\text{Sh-F}} = \{10, 11, 120, 2010, 2011, 0, 121, 122, 20120, 200\}$

$L(\mathcal{C}, \mathcal{S}) = 1.6620 \qquad L_r(\mathcal{S}) = 1.5736 \qquad L(\mathcal{C}_{\text{Sh-F}}, \mathcal{S}) = 1.8770$

Huffman Codes: Example (II)



$\mathcal{C} = \{2, 11, 102, 1002, 10010, 0, 12, 1000, 10011, 101\}$

$\mathcal{C}_{\text{Sh-F}} = \{10, 11, 120, 2010, 2011, 0, 121, 122, 20120, 200\}$

$L(\mathcal{C}, \mathcal{S}) = 1.6620 \qquad L_r(\mathcal{S}) = 1.5736 \qquad L(\mathcal{C}_{\text{Sh-F}}, \mathcal{S}) = 1.8770$

There is still one available node

Outline

Shannon's Entropy of a Discrete Random Variable (I)

It measures the average amount of information in a random variable.

Definition (Shannon's Entropy)

Let X be a random variable with range a finite set $\mathcal{X}$.

$$H(X) = -\sum_{x \in \mathcal{X}} \Pr[X = x] \log_2 \Pr[X = x]$$

Shannon's Entropy of a Discrete Random Variable (I)

It measures the average amount of information in a random variable.

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$$H(X) = -\sum_{x \in \mathcal{X}} \Pr[X = x] \log_2 \Pr[X = x]$$

Relationship with optimal codes:

$$L_r(\mathcal{S}) = H(\mathcal{S}) / \log_2 r$$

"The minimum possible average length for a lossless code for a source $\mathcal{S}$ is actually the average amount of (r-ary) information per source symbol."

Shannon's Entropy of a Discrete Random Variable (II)

Properties of Shannon's entropy:

- **Main inequality:** $H(X,Y,Z) + H(Z) \leq H(X,Z) + H(Y,Z)$ (equality holds only if $\forall z \in \mathcal{Z}$, conditioned to $Z = z$, X and Y are independent) ▸ details...
- $0 \leq H(X) \leq \log_2 |\mathcal{X}|$ (r.h.s. equality holds only for uniform X, and l.h.s. holds for a trivial X)
- $H(X) \leq H(X,Y) \leq H(X) + H(Y)$ (r.h.s. equality holds only for independent X, Y, and l.h.s. holds only if $Y = g(X)$)
- $H(g(X)) \leq H(X)$ (equality holds only for injective g)
- $H(X, g(X,Y)) \leq H(X,Y)$ (equality holds only if $y \mapsto g(x,y)$ is injective $\forall x \in \mathcal{X}$)
- $H(X, g(X)) = H(X)$

Shannon's Entropy of a Discrete Random Variable (II)

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- $H(X,g(X)) = H(X)$

Conditional Entropy

Definition (Conditional Shannon's Entropy)

Let X, Y be random variables with respective ranges $\mathcal{X}, \mathcal{Y}$.
$H(Y \mid X) = H(X, Y) - H(X)$

It measures the average amount of information still in Y after revealing X.

Conditional Entropy

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Let X, Y be random variables with respective ranges $\mathcal{X}, \mathcal{Y}$.
$H(Y \mid X) = H(X, Y) - H(X)$

It measures the average amount of information still in Y after revealing X.

Equivalently, $H(Y \mid X) = \sum_{x \in \mathcal{X}} \Pr[X = x] H(Y \mid X = x)$ or

$$H(Y \mid X) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \Pr[X = x, Y = y] \log_2 \Pr[Y = y \mid X = x]$$

Conditional Entropy: Properties

Similarly to (non-conditional) Shannon's entropy:

- **Main inequality:**
  $H(Y, Z, T \mid X) + H(T \mid X) \leq H(Y, T \mid X) + H(Z, T \mid X)$
- $0 \leq H(Y \mid X) \leq H(Y)$ (r.h.s. equality holds only for independent X, Y, and l.h.s. holds only if $Y = g(X)$)
- $H(Z \mid X, Y) = H(Y, Z \mid X) - H(Y \mid X)$
- $H(Y \mid X) \leq H(Y, Z \mid X) \leq H(Y \mid X) + H(Z \mid X)$
- $H(g(X, Y) \mid X) \leq H(Y \mid X)$
- $H(Y, g(X, Y, Z) \mid X) \leq H(Y, Z \mid X)$
- $H(Y, g(X, Y) \mid X) = H(Y \mid X)$

Mutual Information

Definition (Mutual Information)

Let X, Y be random variables with respective ranges $\mathcal{X}, \mathcal{Y}$.
$I(Y;X) = H(X) + H(Y) - H(X,Y) = H(Y) - H(Y \mid X)$

It measures the amount of information about Y conveyed by X.

Mutual Information

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Let X, Y be random variables with respective ranges $\mathcal{X}, \mathcal{Y}$.
$I(Y;X) = H(X) + H(Y) - H(X,Y) = H(Y) - H(Y \mid X)$

It measures the amount of information about Y conveyed by X.

Some properties

- $I(Y,Z,T;X) + I(T;X) \leq I(Y,T;X) + I(Z,T;X)$
- $I(X;Y) = I(Y;X)$
- $0 \leq I(Y;X) \leq \min(H(X), H(Y))$

Mutual Information

Definition (Mutual Information)

Let X, Y be random variables with respective ranges $\mathcal{X}, \mathcal{Y}$.
$I(Y;X) = H(X) + H(Y) - H(X,Y) = H(Y) - H(Y \mid X)$

It measures the amount of information about Y conveyed by X.

Some properties

- $I(Y,Z,T;X) + I(T;X) \leq I(Y,T;X) + I(Z,T;X)$
- $I(X;Y) = I(Y;X)$
- $0 \leq I(Y;X) \leq \min(H(X), H(Y))$

Definition (Conditional Mutual Information)

Let X, Y, Z be random variables with respective ranges $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.
$I(Z;Y \mid X) = H(Z \mid X) - H(Z \mid X,Y)$

Codes and Cryptography

Jorge L. Villar

MAMME, Fall 2015

END OF PART III

Shannon's Entropy Main Inequality (Submodularity)

The map $f(t) = t \log_2 t$ is convex in $[0, +\infty)$ (because $f'' > 0$)

By discrete Jensen's inequality, for all $a_i > 0$, $t_i > 0$, $i \in \mathcal{I}$

$$\sum_{i \in \mathcal{I}} a_i t_i = 1 \quad \Rightarrow \quad \sum_{i \in \mathcal{I}} a_i t_i \log_2 t_i \geq -\log_2 \sum_{i \in \mathcal{I}} a_i$$

and the equality implies that all t_i are equal.

The main inequality is proven by letting

$$t_{x,y,z} = \frac{\Pr[X = x, Y = y, Z = z] \Pr[Z = z]}{\Pr[X = x, Z = z] \Pr[Y = y, Z = z]}; \qquad \sum_{x,y,z} a_{x,y,z} t_{x,y,z} = 1$$

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go back...

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$$\sum_{x,y,z} a_{x,y,z} = 1$$

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