Codes and Cryptography

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PART X

Outline

Bounded Adversaries

Bounding adversarial resources:

- **Space** (memory). Not a big deal. Memory is cheap, but read/write operations take time!
- **Time** (running time). It actually depends on the computing device (e.g., parallel processors).

Fix a computational model or machine: **Turing Machine**

Turing Machine

A model for mechanic computation, consisting of

- A finite automaton
  - state information $s \in S \cup \{\texttt{init}, \texttt{halt}\}$
  - transition function f
- A storage device ("tape")
  - a "tape", or a sequence of cells containing '0', '1' or '`blank`'
  - a read/write moving head on a specific cell in the tape

Computation step: $(s, wr, mv) \leftarrow f(s, c)$

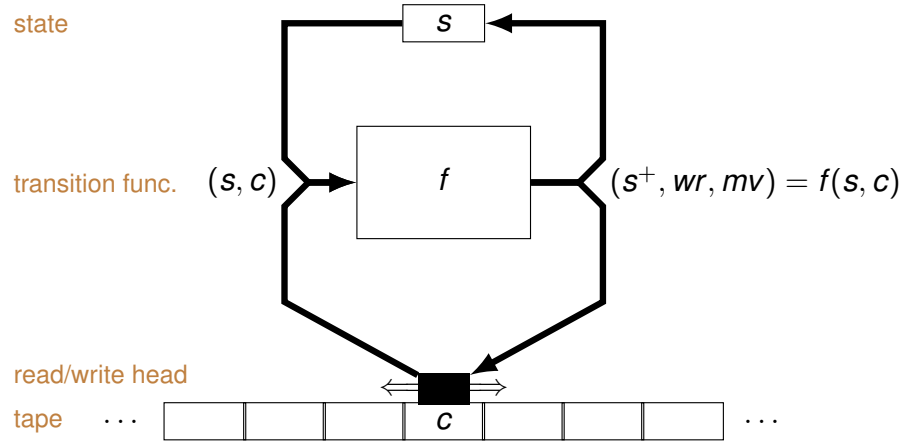
$wr \in \{\texttt{write\_0}, \texttt{write\_1}, \texttt{erase}\}$

$mv \in \{\texttt{move\_left}, \texttt{move\_right}\}$

$c =$ contents of the current cell

Input/Output of the Turing Machine is the content of the tape in the `init`/`halt` state

Turing Machine



$s \in S \cup \{\texttt{init}, \texttt{halt}\}$
$wr \in \{\texttt{write\_0}, \texttt{write\_1}, \texttt{erase}\}$
$mv \in \{\texttt{move\_left}, \texttt{move\_right}\}$

example...

A Sample Turing Machine

Given the binary encoding of a nonnegative integer

$\cdots$ x_0 x_1 $\cdots$ 1 $\cdots$

compute the binary encoding of $y = x + 1$

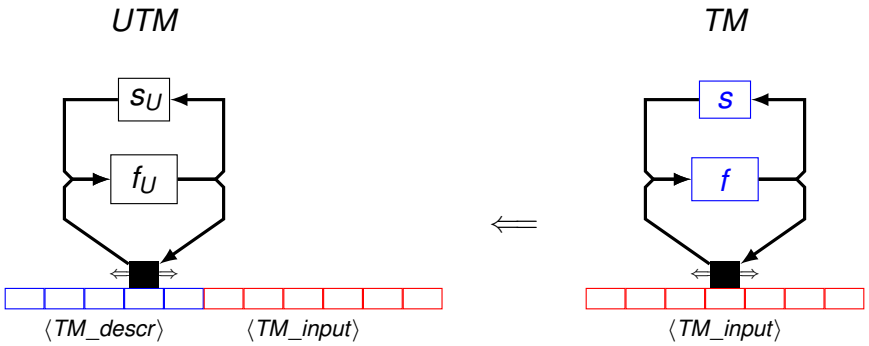
$\cdots$ y_0 y_1 $\cdots$ 1 $\cdots$

IDEA: flip all bits until the first 0 (included), then rewind and halt.

States: $S = \{\texttt{rewind}\}$
Transition function table:
$(\texttt{init}, 0) \mapsto (\texttt{rewind}, \texttt{write\_1}, \texttt{move\_left})$
$(\texttt{init}, 1) \mapsto (\texttt{init}, \texttt{write\_0}, \texttt{move\_right})$
$(\texttt{init}, \texttt{blank}) \mapsto (\texttt{rewind}, \texttt{write\_1}, \texttt{move\_left})$
$(\texttt{rewind}, 0) \mapsto (\texttt{rewind}, \texttt{write\_0}, \texttt{move\_left})$
$(\texttt{rewind}, 1) \mapsto (\texttt{rewind}, \texttt{write\_1}, \texttt{move\_left})$
$(\texttt{rewind}, \texttt{blank}) \mapsto (\texttt{halt}, \texttt{erase}, \texttt{move\_right})$

go back...

Universal Turing Machine



It can simulate any known (classical) computing device with reasonable efficiency

Computational Problems

Parameterized Problem Families: A set $P = \bigcup_{\ell \in \mathbb{Z}^+} P_\ell$ of problem instances parameterized by a size parameter $\ell \in \mathbb{Z}^+$ (e.g., length of the instance description) (Related to the security parameter in cryptography)

- **Search Problems:** Unique correct answer among a large set of possibilities
- **Flexible Problems:** Several correct answers among a large set of possibilities
- **Decision Problems:** The answer is just a single bit (i.e., 'YES' or 'NO')

Decision Problems vs. Search Problems (I)

Most 'natural' problems are stated as Search Problems
. . . but the theory of Decision Problems is cleaner
However, complexity of Decision Problems and of Search Problems can be related

We can describe a Decisional Problem Family P as the set of problem 'YES'-instances (encoded as binary strings):

$L_P = \{x \in P \ : \ \mathrm{sol}(x) = \text{'YES'}\} \subset \{0,1\}^*$

Solving P means checking membership to the **language** L_P

Decision Problems vs. Search Problems (II)

Given a search problem family P, we can define the language

$L_P = \{(x,y) \ : \ x \in P, y \in \mathrm{sol}(x)\}$

Checking membership to L_P is not harder than solving P (if the solution y to x is unique)

REDUCTION: Given (x,y), solve the instance x and check whether $y = \mathrm{sol}(x)$

but sometimes verifying that y solves instance x is easier than directly finding y from x, e.g.,
$P = \{\text{"find a nontrivial factorization of } n\text{"} \ : \ n \in \mathbb{Z}^+\}$,
$L_P = \{(pq,p,q) \ : \ p,q > 1\}$

Problem Hardness (I)

How to measure the efficiency (complexity) of an algorithm?

Running time (computation steps in a Turing Machine)
. . . but it depends on the particular problem instance (input) and on the instance description size ℓ

- **Worst case time** for every fixed ℓ
  Easy to analyze but not always meaningful: e.g. only a few instances taking a huge time
- **Average time**, or expected time for a randomly chosen instance of size ℓ
  Depends on the probability distribution and does not give an upper bound of the time

Problem Hardness (II)

- **Concrete analysis.** Useful to make the best choice of an algorithm solving a specific problem instance, based on the value of ℓ
- **Asymptotic analysis.** Measures how well an algorithm scales with the input size, but cannot be used to compare two algorithms for a precise value of ℓ.

  Meaningful in cryptography, as one can ever choose the size of the keys, with a reasonable impact on efficiency

Uniform vs. Non-Uniform complexity notions

- Uniform means a single algorithm or Turing Machine for all values of ℓ
- Non-Uniform means different algorithms (circuits) can be used for different values of ℓ

Outline

$\mathcal{P}$

The class $\mathcal{P}$: The worst-case running time of the best algorithm **solving all problem instances** in a family $P = \bigcup_{\ell \in \mathbb{Z}^+} P_\ell$ is **upper bounded by a polynomial** in ℓ

NOTATION: **poly** denotes the set of nonnegative functions $f : \mathbb{Z}^+ \to \mathbb{R}$ upper bounded by a polynomial. Addition, multiplication and composition are internal operations.

Definition ($\mathcal{P}$)

A decision problem family P is in $\mathcal{P}$ if there exists a Turing Machine TM such that $\forall x \in P,\ TM(x) = \mathrm{sol}(x)$, and

$$\max_{x \in P_\ell} \mathrm{time}(TM, x) \in \mathbf{poly}(\ell)$$

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$$\max_{x \in P_\ell} \mathrm{time}(TM, x) \in \mathbf{poly}(\ell)$$

Definition ($\mathcal{P}$)

A language L is in $\mathcal{P}$ if there exists a Turing Machine TM such that

$$\forall x \in L,\ TM(x) = 1$$

$$\forall x \in \{0,1\}^* \setminus L,\ TM(x) = 0$$

$$\max_{x \in \{0,1\}^\ell} \mathrm{time}(TM, x) \in \mathbf{poly}(\ell)$$

$\mathcal{P}$

Typical easy computational problems (e.g., basic arithmetic operations) are solved with polynomial-time algorithms

Polynomial-time algorithms form a consistent set, they can be combined in a number of ways producing more polynomial-time algorithms, e.g.

- polynomially long sequence
- polynomially bounded iteration
- composition (subroutines)
- constant depth recursion

$\mathcal{NP}$

The class $\mathcal{NP}$: The worst-case time of the best algorithm solving all **decision** problem 'YES'-instances in P, **with an auxiliary input**, is upper bounded by a polynomial in ℓ

The auxiliary input can be seen as a hint

Definition ($\mathcal{NP}$)

A language L is in $\mathcal{NP}$ if there exists a integer-valued function $q \in$ **poly** and a Turing Machine TM such that

$$\forall x \in \{0,1\}^{\ell},\ x \in L \Leftrightarrow \exists w \in \{0,1\}^{q(\ell)}\ TM(x,w) = 1$$

$$\max_{x \in \{0,1\}^{\ell},\, w \in \{0,1\}^{q(\ell)}} \text{time}(TM, x, w) \in \textbf{poly}(\ell)$$

$\mathcal{NP}$

Most interesting problems in cryptography are in $\mathcal{NP}$

Actually TM just verifies efficiently that some $x \in L$ from a suitable witness w, but there is no witness for $x \in \{0,1\}^* \setminus L$

$L \in$ *co*-$\mathcal{NP} \Leftrightarrow \{0,1\}^* \setminus L \in \mathcal{NP}$

Trivially, $\mathcal{P} \subseteq \mathcal{NP} \cap$ *co*-$\mathcal{NP}$

. . . but it is not known whether $\mathcal{P} = \mathcal{NP}$

$\mathcal{P} = \mathcal{NP}$ would imply that most cryptographic systems are weak!

Randomized Algorithms

A **randomized algorithm** can use random values during the computation (even when the problem instance description is deterministic)

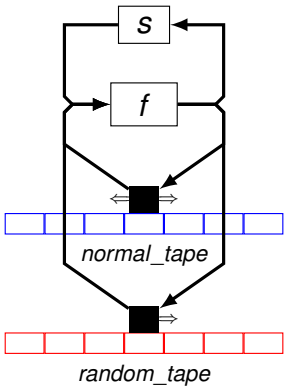
Practice tells us that randomized algorithms can solve problems more efficiently than deterministic ones
. . . but everything in the algorithm (output, running time) can depend on the randomness, and typically we have to tolerate some small error probability

The model: **Probabilistic Turing Machine**

- The same as the plain Turing Machine with an extra read-only tape initialized with random (binary) data
- At every computation step, the current random bit is an input to the transition function

Probabilistic Turing Machine

PTM



Computation step:
$(s, wr, mv, mvr) \leftarrow f(s, c, r)$

$wr \in \{\texttt{write\_0}, \texttt{write\_1}, \texttt{erase}\}$
$mv \in \{\texttt{move\_left}, \texttt{move\_right}\}$
$mvr \in \{\texttt{keep}, \texttt{move\_right}\}$

$c =$ contents of the current normal cell
$r =$ contents of the current random cell

$\mathcal{BPP}$

The class $\mathcal{BPP}$: The worst-case time of the best randomized algorithm **correctly guessing most** decision problem 'YES' and 'NO'-instances in P, is upper bounded by a polynomial in ℓ

Definition ($\mathcal{BPP}$)

A language L is in $\mathcal{BPP}$ if there exists a integer-valued function $p \in$ **poly** and a Probabilistic Turing Machine PTM such that

$$\forall x \in \{0,1\}^\ell,\ x \in L \ \Rightarrow\ \Pr[PTM(x) = 1] > 2/3$$

$$\forall x \in \{0,1\}^\ell,\ x \notin L \ \Rightarrow\ \Pr[PTM(x) = 1] < 1/3$$

$$\forall x \in \{0,1\}^\ell,\ \Pr[\text{time}(PTM, x) < p(\ell)] = 1$$

The thresholds $2/3$ and $1/3$ are arbitrarily fixed

Probability Amplification by Repetition

Given PTM and $\alpha : \mathbb{Z}^+ \to (0, 1/2)$ such that

$$\forall x \in \{0,1\}^\ell,\ x \in L \ \Rightarrow\ \Pr[PTM(x) = 1] > \tfrac{1}{2} + \alpha(\ell)$$

$$\forall x \in \{0,1\}^\ell,\ x \notin L \ \Rightarrow\ \Pr[PTM(x) = 1] < \tfrac{1}{2} - \alpha(\ell)$$

$$\forall x \in \{0,1\}^\ell,\ \Pr[\text{time}(PTM, x) < p(\ell)] = 1$$

we ban build PTM_n for any (odd) $n \in \mathbb{Z}^+$ that on the input of x it runs $PTM(x)$ n times, and decides its output by majority voting.
Then, by Chernoff's bound, details...

$\alpha_n(\ell) > \frac{1}{2} - \left(\sqrt{1 - 4\alpha(\ell)^2}\right)^n$ and $p_n(\ell) = np(\ell)$

In particular,

$\alpha(\ell) = 1/6,\ n > 12\ell \quad \Rightarrow \quad \frac{1}{2} + \alpha_n(\ell) > 1 - 2^{-\ell},\ \frac{1}{2} - \alpha_n(\ell) < 2^{-\ell}$

The same for any $q \in$ **poly** if $\alpha(\ell) = 1/q(\ell)$ and $n > \frac{3}{8}\ell(q(\ell))^2$

Probability Amplification: Details

Fix $x \in \{0,1\}^\ell \cap L$ and define $V_i = TM(x)$ for the i-th run.
$V_i \in \{0,1\}$ and $\Pr[V_i = 1] = 1/2 + \epsilon_x$ for some $\epsilon_x \in (\alpha(\ell), 1/2]$.
Moreover, $V_1, \ldots, V_n$ are independent. Let $V = V_1 + \ldots + V_n$.
For any $t > 0$, by Markov's inequality

$$\Pr[V \le n/2] = \Pr\left[e^{t(n/2-V)} \ge 1\right] \le \mathrm{E}\left[e^{t(n/2-V)}\right] = \mathrm{E}\left[e^{t(1/2-V_1)}\right]^n =$$

$$= \left(\left(\tfrac{1}{2} - \epsilon_x\right) e^{t/2} + \left(\tfrac{1}{2} + \epsilon_x\right) e^{-t/2}\right)^n$$

Therefore $\Pr[V \le n/2] \le \inf_{t>0} \left(\frac{1-2\epsilon_x}{2} e^{t/2} + \frac{1+2\epsilon_x}{2} e^{-t/2}\right)^n$

The minimum value is attained when $e^t = \frac{1+2\epsilon_x}{1-2\epsilon_x}$ and then

$$\Pr[V > n/2] \ge 1 - \left(\sqrt{1 - 4\epsilon_x^2}\right)^n \ge 1 - \left(\sqrt{1 - 4\alpha(\ell)^2}\right)^n$$

go back...

Bounded Computational Power

NOTATION: **negl** denotes the set of nonnegative functions $f : \mathbb{Z}^+ \to \mathbb{R}$ vanishing faster than the inverse of any nonzero polynomial

$$f \in \textbf{negl} \ \Leftrightarrow\ \forall c \in \mathbb{Z}^+ \lim_{t \to +\infty} f(t) t^{-c} = 0$$

Addition, multiplication and multiplication by functions in **poly** preserve negligible functions

Bounded Computational Power

The previous considerations motivate the following definitions for infeasible computation:

Definition (Hard-on-Average Search Problem)

A search problem is considered hard if any Probabilistic Turing Machine can solve a **random** problem instance in polynomial time only with a **negligible probability**

Definition (Hard-on-Average Decision Problem)

A decision problem is considered hard if any Probabilistic Turing Machine can solve a **random** problem instance in polynomial time only with a **negligible advantage** with respect to a random guess

NOTE: feasible $\neq$ efficient

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END OF PART X