CRYE 6127 Introduction to Cryptology

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Public Key Encryption

Outline

Cryptography in a Multiuser Setting

Private Storage: One secret key per user

Private Communication: One shared secret key per channel

In a Private Communication Network
n users $\Rightarrow$ $\binom{n}{2}$ secret keys
Every user stores $n-1$ independent secret keys!

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n users $\Rightarrow$ $\binom{n}{2}$ secret keys
Every user stores $n-1$ independent secret keys!

Public Key Cryptography: Every user generates a key pair (pk, sk), publishes pk and keeps sk secret

Every user stores only one secret key . . . but public keys must be reliably distributed

The Diffie-Hellman Key Agreement Protocol

The seminal work in public key cryptography!

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of prime order q.

$$G = \{g^0 = 1, g, g^2, g^3, \ldots, g^{q-1}\}, \qquad g^q = 1$$

E.g. g is a nontrivial solution of $g^q = 1 \pmod{p}$.

p, q are large primes and q divides $p - 1$.

All multiplications and powers are modulo p.

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Small example (insecure!):
$p = 23$, $q = 11$, $g = 2$ as $2^{11} = 2048 = 23 * 89 + 1$.

The Diffie-Hellman Key Agreement Protocol

Party $P_i(G, q, g)$:

$x_i \leftarrow \{0, \ldots, q-1\}$

publish $y_i = g^{x_i}$

wait for y_j

$k_{ij} = y_j^{x_i}$ // common key for parties P_i and P_j

Common key: $k_{ij} = y_i^{x_j} = y_j^{x_i} = g^{x_i x_j}$.

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Assumption (Decision Diffie-Hellman (DDH))

(Informal) Given only G, g, y_i and y_j, the common key k_{ij} is indistinguishable from random.

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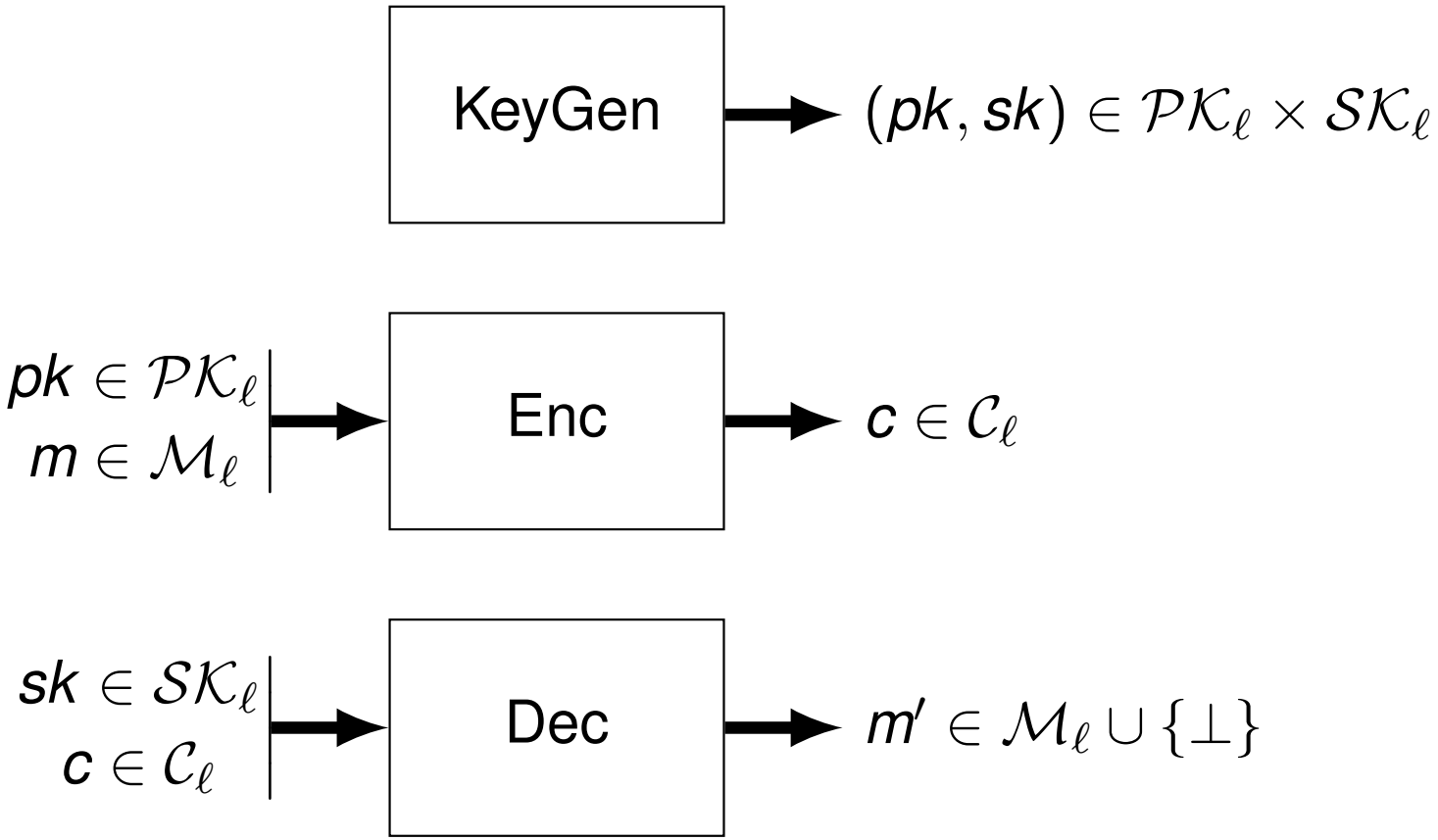
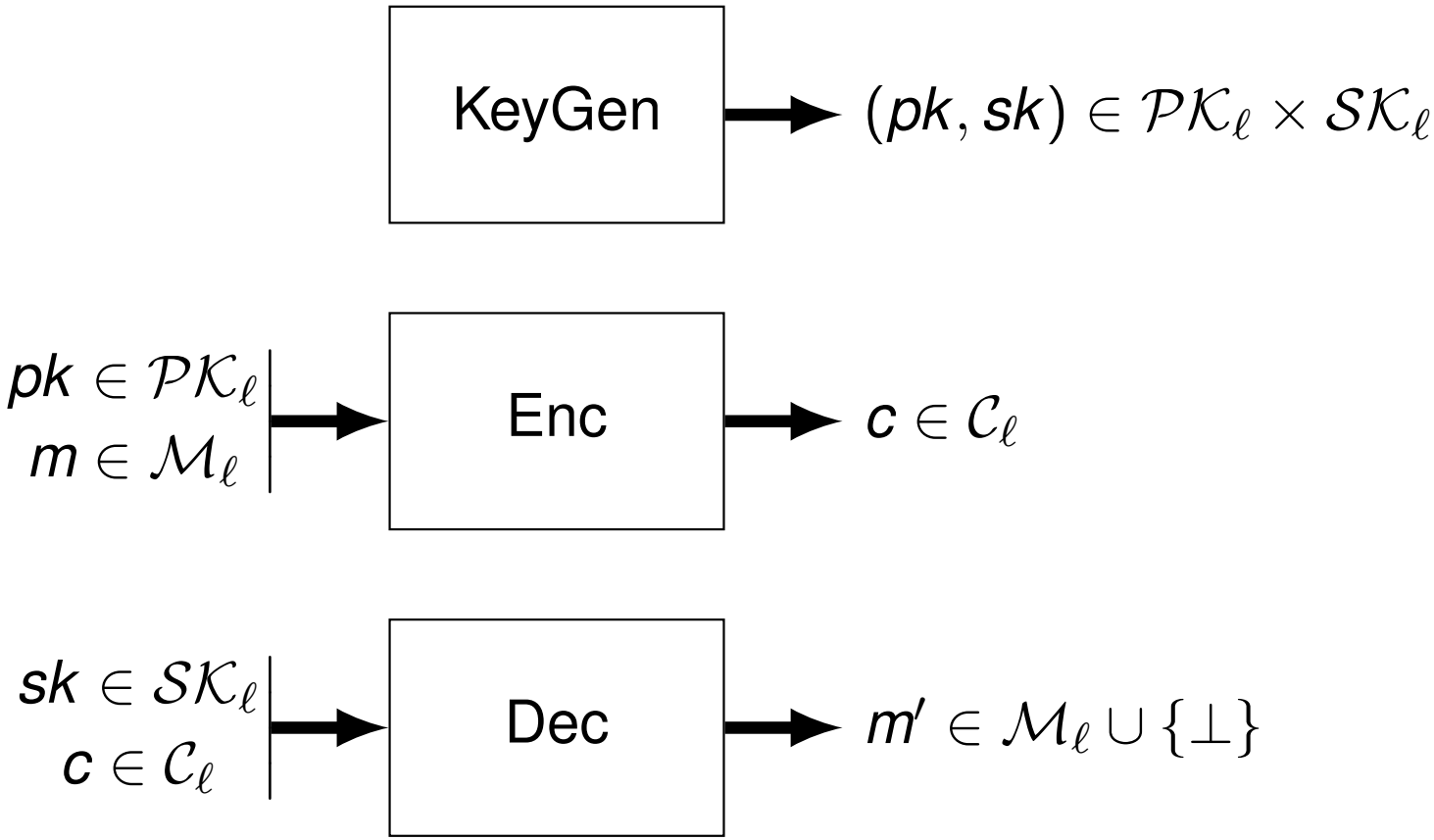
Combining Diffie-Hellman key agreement protocol with the one-time pad, we can build a **Public-Key Encryption Scheme**.

Public Key Encryption: Syntax



Public Key Encryption: Correctness



$$\forall m \in \mathcal{M}_\ell \ \forall (pk, sk) \leftarrow \mathsf{KeyGen}(\ell), \quad m = \mathsf{Dec}(sk, \mathsf{Enc}(pk, m))$$

The Man-in-the-Middle Attack

A	B
send(B, pk_A);	**receive**(pk_A)
	$c_B \leftarrow \mathrm{Enc}(pk_A, m_B)$;
receive(c_B)	**send**(A, c_B);
$m_B \leftarrow \mathrm{Dec}(sk_A, c_B)$;	

The Man-in-the-Middle Attack

In a multiparty scenario, the adversary C can impersonate a user by replacing the public key

A	C	B
send(B, pk_A);	**intercept**; **send**(B, pk');	**receive**(pk')
		$c_B \leftarrow \text{Enc}(pk', m_B)$; **send**$(A, c_B)$;
	intercept; $m_B \leftarrow \text{Dec}(sk', c_B)$; $c' \leftarrow \text{Enc}(pk_A, m')$; **send**$(A, c')$;	
receive(c') $m' \leftarrow \text{Dec}(sk_A, c')$;		

The Man-in-the-Middle Attack

Public keys need to be **certified**

A	C	B
send(B, pk_A);	**intercept**; **send**(B, pk');	**receive**(pk') **verify**(A, pk');
		$c_B \leftarrow \mathrm{Enc}(pk', m_B)$; **send**$(A, c_B)$;
	intercept; $m_B \leftarrow \mathrm{Dec}(sk', c_B)$; $c' \leftarrow \mathrm{Enc}(pk_A, m')$; **send**$(A, c')$;	
receive(c') $m' \leftarrow \mathrm{Dec}(sk_A, c')$;		

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- **Identity Based Cryptography:** Public keys are just the users' identities, and then they are not replaceable (e.g., they are just the users' names or their email addresses)

The Man-in-the-Middle Attack

Public keys need to be **certified**

- **Public Key Infrastructure:** Public keys are validated by a trusted entity
- **Identity Based Cryptography:** Public keys are just the users' identities, and then they are not replaceable (e.g., they are just the users' names or their email addresses) . . . but then a Key Generation Center generates and then knows all the secret keys

Outline

ElGamal PKE Scheme

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of ℓ bits long prime order q. Set $\mathcal{M} = G$ and $\mathcal{C} = G \times G$.

InstGen(ℓ) : // generates some public parameters
  param $\leftarrow (G, q, g)$

KeyGen(param) :
  $x \leftarrow \mathbb{Z}_q$
  $y \leftarrow g^x$
  output (y, x)

Enc(param, y, m) :
  $r \leftarrow \mathbb{Z}_q$ // probabilistic encryption
  output $(g^r, y^r m)$ // Diffie-Hellman Key & One-Time Pad

Dec(param, $x, (c_1, c_2)$) :
  output $c_2 c_1^{-x}$ // encryption randomness is removed

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Correctness:

$c_2 c_1^{-x} = y^r m (g^r)^{-x} = g^{rx} m g^{-rx} = m.$

Hashed ElGamal PKE Scheme

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of ℓ bits long prime order q. Set $\mathcal{M} = \{0,1\}^\ell$ and $\mathcal{C} = G \times \{0,1\}^\ell$.

InstGen(ℓ) : // generates some public parameters
  param $\leftarrow (G, q, g, H)$ // H is a hash function

KeyGen(param) :
  $x \leftarrow \mathbb{Z}_q$
  $y \leftarrow g^x$
  output (y, x)

Enc(param, y, m) :
  $r \leftarrow \mathbb{Z}_q$ // probabilistic encryption
  output $(g^r, H(y^r) \oplus m)$ // Diffie-Hellman Key & One-Time Pad

Dec(param, x, (c_1, c_2)) :
  output $c_2 \oplus H(c_1^x)$ // encryption randomness is removed

Correctness:

$c_2 \oplus H(c_1^x) = H(y^r) \oplus m \oplus H(g^{rx}) = H(g^{rx}) \oplus m \oplus H(g^{rx}) = m$.

ElGamal Variants and Security

Definition (DLOG Problem)

Given G, q, g and a random element $y \in G$, find x such that $y = g^x$.

If DLOG is easy, ElGamal is not secure (the secret key can be obtained form the public information).

But hardness of DLOG is not enough for security of ElGamal!

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Known ElGamal implementations:

- Subgroups of $GF(p)^\times$ $|p| \approx 2048$ bits, $|q| \approx 224$ bits
- Subgroups of the group of points of an elliptic curve over $GF(p)$ $|p| \approx 224$ bits, $|q| \approx 224$ bits
- Other less common structures (Jacobian varieties, non-abelian groups, ...)

ElGamal PKE Scheme (Elliptic Curve Version)

Let $G = \langle B \rangle$ be a cyclic (additive) group of ℓ bits long prime order q. Set $\mathcal{M} = G$ and $\mathcal{C} = G \times G$.

InstGen(ℓ) : // generates some public parameters
  param $\leftarrow (G, q, B)$

KeyGen(param) :
  $x \leftarrow \mathbb{Z}_q$
  $Y \leftarrow xB$
  output (Y, x)

Enc(param, Y, M) :
  $r \leftarrow \mathbb{Z}_q$ // probabilistic encryption
  output $(rB, rY + M)$ // Diffie-Hellman Key & One-Time Pad

Dec(param, $x, (C_1, C_2)$) :
  output $C_2 - xC_1$ // encryption randomness is removed

Correctness:

$C_2 - xC_1 = rY + M - xrB = rxB + M - xrB = M.$

RSA PKE Scheme

KeyGen(ℓ) :
  Choose random $\ell/2$ bits long primes p, q
  and $e \geq 3$ coprime with $\phi(n) = (p-1)(q-1)$.
  $n \leftarrow pq$
  Set $\mathcal{M} = \mathcal{C} = \mathbb{Z}_n^\times$.
  $d \leftarrow e^{-1} \mod \text{lcm}(p-1, q-1)$
  output $((n, e), (n, d))$ // factoring n must be hard!

Enc(n, e, m) :
  output $m^e \mod n$

Dec(n, d, c) :
  output $c^d \mod n$

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Correctness:

$c^d \mod n = (m^e)^d \mod n = m^{ed} \mod n = m$,

because $m^{\phi(n)} \mod n = 1$ (Fermat's little theorem)

RSA Variants and Security

Definition (Integer Factoring Problem)

Given $n = pq$, for p, q primes of the same length, find p or q.

If Factoring is easy, RSA is not secure (the secret key can be obtained form the public information).

Typical size of p and q is 1024 bits.

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Two main ways to generate the public key:

- The public exponent e is random and coprime with $\phi(n) = (p-1)(q-1)$.
- The public exponent e is fixed and $p-1$ and $q-1$ are coprime with e.
  Typical values $e = 17$ or 65537.

Rabin PKE Scheme

KeyGen(ℓ) :

Choose random $\ell/2$ bits long primes p, q such that $p, q = 3 \mod 4$.

$n \leftarrow pq$

Set $\mathcal{M} = \mathcal{C} = QR_n$. // the set of quadratic residues mod n

output $((n, e), (p, q))$ // factoring n must be hard!

Enc(n, m) :

output $m^2 \mod n$

Dec(p, q, c) :

$m_p = c^{(p+1)/4} \mod p$; $m_q = c^{(q+1)/4} \mod q$

Use Chineese Remainder Theorem to compute m from (m_p, m_q).

output m

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Correctness:

Since $m \in QR_n$ then $m = x^2 \mod n$

$$m_p = c^{(p+1)/4} = m^{(p+1)/2} = x^{p+1} = x^2 = m \pmod p$$

$$m_q = c^{(q+1)/4} = m^{(q+1)/2} = x^{q+1} = x^2 = m \pmod q$$

Paillier PKE Scheme

KeyGen(ℓ) :

Choose random $\ell/2$ bits long primes p, q.

$n \leftarrow pq$

Set $\mathcal{M} = \mathbb{Z}_n$ and $\mathcal{C} = \mathbb{Z}_{n^2}^{\times}$.

$\lambda \leftarrow \text{lcm}(p-1, q-1)$

output $(n, (n, \lambda))$ // factoring n must be hard!

Enc(n, m) :

$r \leftarrow \mathbb{Z}_n^{\times}$

output $(1 + mn)r^n \mod n^2$

Dec(n, λ, c) : // RSA with $e = n$ gives another way to decrypt c

$c' \leftarrow c^{\lambda} \mod n^2$

output $\frac{c'-1}{n}\lambda^{-1} \mod n$

Paillier PKE Scheme

Correctness:

$c' = c^{\lambda} \mod n^2 = (1+mn)^{\lambda} r^{n\lambda} \mod n^2 = 1 + \lambda mn \mod n^2.$

Then, $\frac{c'-1}{n} = \frac{\lambda mn \mod n^2}{n} = m\lambda \mod n$

and $\frac{c'-1}{n}\lambda^{-1} \mod n = m.$

Paillier PKE Scheme

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Then, $\frac{c'-1}{n} = \frac{\lambda mn \mod n^2}{n} = m\lambda \mod n$

and $\frac{c'-1}{n}\lambda^{-1} \mod n = m.$

Another way to decrypt: Use RSA with $e = n$

$c^d \mod n = ((1+mn)r^n)^d \mod n = r^{nd} \mod n = r.$

Once r is recovered, then

$$\frac{(cr^{-n}-1) \mod n^2}{n} = \frac{((1+mn)-1) \mod n^2}{n} = m.$$

Paillier PKE Scheme

Definition (Integer Factoring Problem)

Given $n = pq$, for p, q primes of the same length, find p or q.

If Factoring is easy, Paillier is not secure (the secret key can be obtained form the public information).

Typical size of p and q is 1024 bits.

Paillier PKE Scheme

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Given $n = pq$, for p, q primes of the same length, find p or q.

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Typical size of p and q is 1024 bits.

Variant: $1 + nm$ can be replaced by $g^m \mod n^2$, for some special values of g.

Encryption is now $c = g^m r^n \mod n^2$

Decryption uses the fact that $g^\lambda \mod n^2 = 1 + \beta n$ for some β.

Regev PKE

InstGen(ℓ) :
Select a prime q and integers n, k.
Select a Gaussian distribution $N(0, \sigma^2)$.
Set $\mathcal{M} = \{0, 1\}$.
param $\leftarrow (q, n, k, \sigma)$

KeyGen(ℓ) :
$A \leftarrow \mathbb{Z}_q^{k \times n}$; $\mathbf{s} \leftarrow \mathbb{Z}_q^{n \times 1}$; $\mathbf{x} \leftarrow \mathbb{Z}_q^{k \times 1}$;
where $x_i \leftarrow \mathbf{round}(qN(0, \sigma^2)) \mod q$
$\mathbf{output}((A, \mathbf{b} = A\mathbf{s} + \mathbf{x} \mod q), \mathbf{s})$

Enc(param, $A, \mathbf{b}, m$) :
$\mathbf{r} \leftarrow \{0, 1\}^{1 \times k}$
$\mathbf{output}$ $(\mathbf{c_1}, c_2) = (\mathbf{r}A \mod q, \mathbf{r} \cdot \mathbf{b} + m\frac{q-1}{2} \mod q)$

Dec(param, $\mathbf{s}, (\mathbf{c_1}, c_2)$) :
$\mu = c_2 - \mathbf{c_1} \cdot \mathbf{s} \mod q$
Decide m by proximity of μ to $\{0, \frac{q-1}{2}\}$
$\mathbf{output}$ m

Regev PKE

Correctness:

$$\mu = c_2 - \mathbf{c_1} \cdot \mathbf{s} \mod q = \mathbf{r} \cdot \mathbf{b} + m\frac{q-1}{2} - \mathbf{r}A \cdot \mathbf{s} \mod q =$$
$$\mathbf{r} \cdot (A\mathbf{s} + \mathbf{x}) + m\frac{q-1}{2} - \mathbf{r}A \cdot \mathbf{s} \mod q = \mathbf{r} \cdot \mathbf{x} + m\frac{q-1}{2} \mod q.$$

$\mathbf{r} \cdot \mathbf{x}$ is small (Depends on k and σ) $\Rightarrow$ $\mu \approx m\frac{q-1}{2}$.

There is a (tiny) positive probability of decryption error.

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$\mathbf{r} \cdot (A\mathbf{s} + \mathbf{x}) + m\frac{q-1}{2} - \mathbf{r}A \cdot \mathbf{s} \mod q = \mathbf{r} \cdot \mathbf{x} + m\frac{q-1}{2} \mod q.$

$\mathbf{r} \cdot \mathbf{x}$ is small (Depends on k and σ) $\Rightarrow \quad \mu \approx m\frac{q-1}{2}$.

There is a (tiny) positive probability of decryption error.

Typical choice for parameters:

- $n^2 < q < 2n^2$, q prime
- $k = (1+\epsilon)(n+1)\log q$ for some ϵ
- $\sigma = o(1/\sqrt{n}\log n)$

Security depends on some lattice theory problems (no efficient quantum attack is known!).

Summary of Constructions

- ElGamal is based on DLOG, Regev uses lattices and RSA, Rabin and Paillier are based on Factoring
- ElGamal can be based on elliptic curves for shorter keysizes
- All except Regev are broken with a quantum computer
- RSA and Rabin are deterministic, while the others are probabilistic
- The ratio of ciphertext/plaintext sizes is 1 for RSA and Rabin, 2 for ElGamal and Paillier, and it is huge for Regev

Outline

Homomorphic Encryption

Let $\Pi = (\text{KeyGen}, \text{Enc}, \text{Dec})$ be a PKE scheme with message space $\mathcal{M}$ and ciphertext space $\mathcal{C}$. Let $\otimes$ be a binary operation defined on $\mathcal{M}$.

Definition (Weakly Homomorphic Encryption)

Π is weakly homomorphic with respect to $\otimes$ if there exists a an efficient algorithm $\text{HomEval}(\cdot)$ such that for all properly generated (pk, sk) and all messages $m_1, m_2 \in \mathcal{M}$

$$\text{Dec}(sk, \text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))) = m_1 \otimes m_2$$

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$$\text{Dec}(sk, \text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))) = m_1 \otimes m_2$$

It means computing on encrypted data without decrypting it!

Examples

Typically, there exist efficient group operations, say $\otimes$ in $\mathcal{M}$ and $\odot$ on $\mathcal{C}$, such that

$$\mathsf{Dec}(sk, c_1 \odot c_2) = \mathsf{Dec}(sk, c_1) \otimes \mathsf{Dec}(sk, c_2)$$

and one can just define $\mathsf{HomEval}(pk, c_1, c_2) = c_1 \odot c_2$ to obtain a weak homomorphic PKE.

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Example: RSA is homomorphic w.r.t. multiplication in $\mathbb{Z}_n^\times$

$$\mathrm{Dec}(n, d, c_1 c_2 \mod n) = (c_1 c_2)^d \mod n = c_1^d c_2^d \mod n =$$
$$= \mathrm{Dec}(n, d, c_1)\mathrm{Dec}(n, d, c_2) \mod n$$

or

$$\mathrm{Enc}(n, e, m_1)\mathrm{Enc}(n, e, m_2) \mod n = m_1^e m_2^e \mod n =$$
$$= (m_1 m_2)^e \mod n = \mathrm{Enc}(n, e, m_1 m_2 \mod n)$$

Strongly Homomorphic Encryption

Weak homomorphism can leak some information about

- whether a ciphertext c is computed via HomEval or directly with Enc,
- whether certain ciphertext c_1 was used in a call to HomEval.

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Definition (Strongly Homomorphic Encryption)

Π is strongly homomorphic if in addition, for any valid (pk, sk), m_1 and m_2, the random variables $\text{Enc}(pk, m_1 \otimes m_2)$ and $\text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))$ are independent and identically distributed.

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Lemma

Any weakly homomorphic deterministic PKE is also strongly.

Ciphertext Rerandomization

In a strongly homomorphic probabilistic PKE, HomEval(.) needs to remove any correlation between the randomness in $\mathrm{Enc}(pk, m_1)$, $\mathrm{Enc}(pk, m_2)$ and $\mathrm{Enc}(pk, m_1 \otimes m_2)$.

Main idea: Define $\mathrm{ReRand}(pk, \cdot)$, so that it transforms any ciphertext c into another one c' such that

- $\mathrm{Dec}(sk, c) = \mathrm{Dec}(sk, c')$,
- c and c' have independent randomness,
- c' has the same probability distribution as $\mathrm{Enc}(pk, m)$.

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E.g., $\mathrm{ReRand}(pk, c) = \mathrm{WeakHomEval}(pk, c, \mathrm{Enc}(pk, m_0))$, where m_0 is the neutral element (0 for addition, 1 for multiplication)

Then, we simply define:

$$\mathrm{HomEval}(pk, c_1, c_2) = \mathrm{ReRand}(pk, \mathrm{WeakHomEval}(pk, c_1, c_2)).$$

Examples

ElGamal is strongly homomorphic in G:

$\mathsf{Enc}(y, m_1) \cdot \mathsf{Enc}(y, m_2) = (g^{r_1}, y^{r_1} m_1) \cdot (g^{r_2}, y^{r_2} m_2) =$
$(g^{r_1+r_2}, y^{r_1+r_2} m_1 m_2)$ is an encryption of $m_1 m_2$.

$\mathsf{HomEval}(y, c_1, c_2) = c_1 \cdot c_2 \cdot (g^{r_3}, y^{r_3})$ for a random $r_3 \in \mathbb{Z}_q$.

It results in $(g^r, y^r m_1 m_2)$ for $r = r_1 + r_2 + r_3$, which is independent of r_1 and r_2.

Examples

Paillier is (additive) strongly homomorphic in $\mathbb{Z}_n$:

$\mathsf{Enc}(n, m_1)\mathsf{Enc}(n, m_2) \mod n^2 = (1 + m_1 n) r_1^n (1 + m_2 n) r_2^n \mod n^2 = (1 + (m_1 + m_2)n)(r_1 r_2)^n \mod n^2$ is an encryption of $m_1 + m_2 \mod n$.

$\mathsf{HomEval}(n, c_1, c_2) = c_1 c_2 r_3^n \mod n^2$ for a random $r_3 \in \mathbb{Z}_n^\times$.

It results in $(1 + (m_1 + m_2)n) r^n \mod n^2$ for $r = r_1 r_2 r_3 \mod n$, which is independent of r_1 and r_2.

Homomorphic Encryption Summary

- RSA is homomorphic w.r.t. multiplication $\mod n$, for $n = pq$.
- ElGamal is homomorphic w.r.t. the operation in the group G.
- Hashed ElGamal has no homomorphic property due to the hash function.
- Paillier is homomorphic w.r.t. addition $\mod n$, for $n = pq$.
- Regev is weakly homomorphic w.r.t. addition $\mod 2$ (XOR operation).

A client application would be interested in common logical or arithmetic operations on bitstrings (none of the above are!).

Applications

If $n = pq$ is large enough, Paillier is quite homomorphic for normal addition of positive integers $\Rightarrow$ Lots of applications!

It can also be done with ElGamal and exponentiated messages $m = g^x$. Then, $g^{x_1} g^{x_2} = g^{x_1 + x_2}$, but decryption (i.e. recovering x) can only be done if x is small.

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Example: (very simplistic) electronic voting

- Encrypted ballot: $c_i = \text{Enc}(pk, ballot)$ where $ballot \in \{0, 1\}$
- Aggregate ballot: $c \leftarrow \text{HomEval}(pk, c_1, \cdots, c_n)$
- Tally computation: $tally = \text{Dec}(sk, c)$, where $tally = \sum_i ballot = \#\{ballot = 1\}$

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Many verifications should be added for a realistic voting scheme (detection of corrupted ballots, multiple voting, corrupted tally...) while preserving voter's privacy.

Fully Homomorphic Encryption

A **fully homomorphic** PKE scheme has homomorphic properties for more than one operation (e.g., addition and multiplication).

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Arithmetic circuits over the ring $\mathbb{Z}_n$ include boolean circuits, because if one encodes logical '0' and '1' as arithmetic '0' and '1', then

- x_1 **AND** $x_2 = x_1 x_2$
- x_1 **OR** $x_2 = x_1 + x_2 - x_1 x_2$
- x_1 **XOR** $x_2 = x_1 + x_2 - 2x_1 x_2$
- **NOT** $x = 1 - x$

CRYE 6127 Introduction to Cryptology

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UBa Cyber Crypto Center, Fall 2025

Public Key Encryption

—END—