

Data Protection

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Outline

- 1 Public-Key Encryption
- 2 PKE Practical Constructions
- 3 Homomorphic Encryption

Cryptography in a Multiuser Setting

Private Storage: One secret key per user

Private Communication: One shared secret key per channel

In a Private Communication Network

n users $\Rightarrow \binom{n}{2}$ secret keys

Every user stores $n - 1$ independent secret keys!

The Diffie-Hellman Key Agreement Protocol

The seminal work in public key cryptography!

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of prime order q .

$$G = \{g^0 = 1, g, g^2, g^3, \dots, g^{q-1}\}, \quad g^q = 1$$

E.g. g is a nontrivial solution of $g^q = 1 \pmod{p}$.

p, q are large primes and q divides $p - 1$.

All multiplications and powers are modulo p .

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All multiplications and powers are modulo p .

Small example (insecure!):

$$p = 23, q = 11, g = 2 \quad \text{as} \quad 2^{11} = 2048 = 23 * 89 + 1.$$

The Diffie-Hellman Key Agreement Protocol

Party $P_i(G, q, g)$:

$x_i \leftarrow \{0, \dots, q-1\}$

publish $y_i = g^{x_i}$

wait for y_j

$k_{ij} = y_j^{x_i}$ // common key for parties P_i and P_j

Common key: $k_{ij} = y_i^{x_j} = y_j^{x_i} = g^{x_i x_j}$.

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Assumption (Decision Diffie-Hellman (DDH))

(Informal) Given only G, g, y_i and y_j , the common key k_{ij} is indistinguishable from random.

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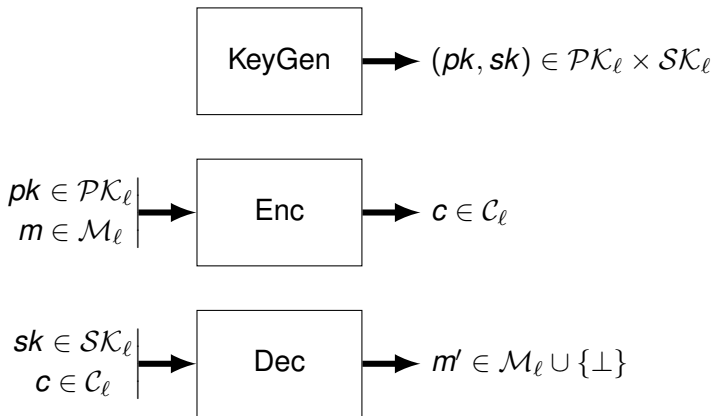
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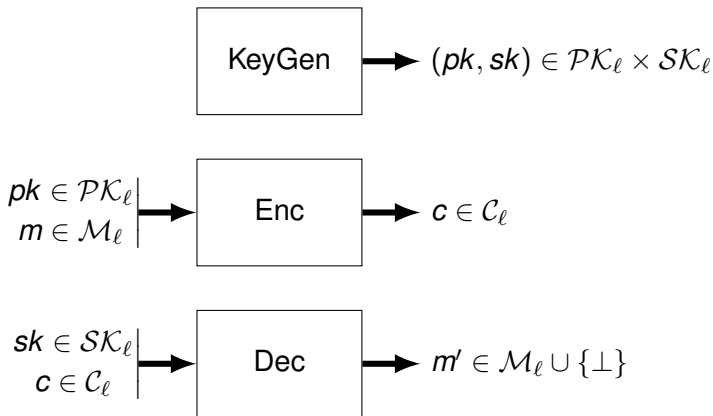
(Informal) Given only G, g, y_i and y_j , the common key k_{ij} is indistinguishable from random.

Combining Diffie-Hellman key agreement protocol with the one-time pad, we can build a **Public-Key Encryption Scheme**.

Public Key Encryption: Syntax



Public Key Encryption: Correctness



$$\forall m \in \mathcal{M}_\ell \quad \forall (pk, sk) \leftarrow \text{KeyGen}(\ell), \quad m = \text{Dec}(sk, \text{Enc}(pk, m))$$

The Man-in-the-Middle Attack

A

B

send(B, pk_A);

receive(pk_A)

receive(c_B)

 $c_B \leftarrow \text{Enc}(pk_A, m_B);$
send(A, c_B);

 $m_B \leftarrow \text{Dec}(sk_A, c_B);$

The Man-in-the-Middle Attack

In a multiparty scenario, the adversary C can impersonate a user by replacing the public key

A

C

B

send (B, pk_A) ;

intercept;
send (B, pk') ;

receive (pk')

receive (c')
 $m' \leftarrow \text{Dec}(sk_A, c')$;

intercept;
 $m_B \leftarrow \text{Dec}(sk', c_B)$;
 $c' \leftarrow \text{Enc}(pk_A, m')$;
send (A, c') ;

$c_B \leftarrow \text{Enc}(pk', m_B)$;
send (A, c_B) ;

The Man-in-the-Middle Attack

Public keys need to be **certified**

A

C

B

send(B, pk_A);

intercept;
send(B, pk');

receive(pk')
verify(A, pk');

receive(c')
 $m' \leftarrow \text{Dec}(sk_A, c')$;

intercept;
 $m_B \leftarrow \text{Dec}(sk', c_B)$;
 $c' \leftarrow \text{Enc}(pk_A, m')$;
send(A, c');

$c_B \leftarrow \text{Enc}(pk', m_B)$;
send(A, c_B);

The Man-in-the-Middle Attack

Public keys need to be **certified**

- **Public Key Infrastructure:** Public keys are validated by a trusted entity
- **Identity Based Cryptography:** Public keys are just the users' identities, and then they are not replaceable (e.g., they are just the users' names or their email addresses)

The Man-in-the-Middle Attack

Public keys need to be **certified**

- **Public Key Infrastructure:** Public keys are validated by a trusted entity
- **Identity Based Cryptography:** Public keys are just the users' identities, and then they are not replaceable (e.g., they are just the users' names or their email addresses) . . . but then a Key Generation Center generates and then knows all the secret keys

EIGamal PKE Scheme

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of ℓ bits long prime order q . Set $\mathcal{M} = G$ and $\mathcal{C} = G \times G$.

InstGen(ℓ) : // generates some public parameters
param $\leftarrow (G, q, g)$

KeyGen(**param**) :
 $x \leftarrow \mathbb{Z}_q$
 $y \leftarrow g^x$
output (y, x)

Enc(**param**, y , m) :
 $r \leftarrow \mathbb{Z}_q$ // probabilistic encryption
output $(g^r, y^r m)$ // Diffie-Hellman Key & One-Time Pad

Dec(**param**, x , (c_1, c_2)) :
output $c_2 c_1^{-x}$ // encryption randomness is removed

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KeyGen(param) :

$$x \leftarrow \mathbb{Z}_q$$

$$y \leftarrow g^x$$

output (y, x)

Enc(param, y, m) :

$$r \leftarrow \mathbb{Z}_q \quad // \text{probabilistic encryption}$$

output $(g^r, y^r m)$ // Diffie-Hellman Key & One-Time Pad

Dec($\text{param}, x, (c_1, c_2)$) :

output $c_2 c_1^{-x}$ // encryption randomness is removed

Correctness:

$$c_2 c_1^{-x} = y^r m (g^r)^{-x} = g^{rx} m g^{-rx} = m.$$

Hashed ElGamal PKE Scheme

Let $G = \langle g \rangle$ be a cyclic (multiplicative) group of ℓ bits long prime order q . Set $\mathcal{M} = \{0, 1\}^\ell$ and $\mathcal{C} = G \times \{0, 1\}^\ell$.

InstGen(ℓ) : // generates some public parameters
 param $\leftarrow (G, q, g, H)$ // H is a hash function

KeyGen(param) :

$x \leftarrow \mathbb{Z}_q$

$y \leftarrow g^x$

output (y, x)

Enc(param, y, m) :

$r \leftarrow \mathbb{Z}_q$ // probabilistic encryption

output $(g^r, H(y^r) \oplus m)$ // Diffie-Hellman Key & One-Time Pad

Dec(param, $x, (c_1, c_2)$) :

output $c_2 \oplus H(c_1^x)$ // encryption randomness is removed

Correctness:

$$c_2 \oplus H(c_1^x) = H(y^r) \oplus m \oplus H(g^{rx}) = H(g^{rx}) \oplus m \oplus H(g^{rx}) = m.$$

ElGamal Variants and Security

Definition (DLOG Problem)

Given G , q , g and a random element $y \in G$, find x such that $y = g^x$.

If DLOG is easy, ElGamal is not secure (the secret key can be obtained from the public information).

But hardness of DLOG is not enough for security of ElGamal!

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Known ElGamal implementations:

- Subgroups of $GF(p)^\times$ $|p| \approx 2048$ bits, $|q| \approx 224$ bits
- Subgroups of the group of points of an elliptic curve over $GF(p)$ $|p| \approx 224$ bits, $|q| \approx 224$ bits
- Other less common structures (Jacobian varieties, non-abelian groups, ...)

EIGamal PKE Scheme (Elliptic Curve Version)

Let $G = \langle B \rangle$ be a cyclic (additive) group of ℓ bits long prime order q . Set $\mathcal{M} = G$ and $\mathcal{C} = G \times G$.

InstGen(ℓ) : // generates some public parameters
param $\leftarrow (G, q, B)$

KeyGen(**param**) :
 $x \leftarrow \mathbb{Z}_q$
 $Y \leftarrow xB$
output (Y, x)

Enc(**param**, Y , M) :
 $r \leftarrow \mathbb{Z}_q$ // probabilistic encryption
output $(rB, rY + M)$ // Diffie-Hellman Key & One-Time Pad

Dec(**param**, x , (C_1, C_2)) :
output $C_2 - xC_1$ // encryption randomness is removed

Correctness:

$$C_2 - xC_1 = rY + M - xrB = rxB + M - xrB = M.$$

RSA PKE Scheme

KeyGen(ℓ) :

Choose random $\ell/2$ bits long primes p, q
and $e \geq 3$ coprime with $\phi(n) = (p-1)(q-1)$.

$n \leftarrow pq$

Set $\mathcal{M} = \mathcal{C} = \mathbb{Z}_n^\times$.

$d \leftarrow e^{-1} \pmod{\text{lcm}(p-1, q-1)}$

output $((n, e), (n, d))$ // factoring n must be hard!

Enc(n, e, m) :

output $m^e \pmod n$

Dec(n, d, c) :

output $c^d \pmod n$

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Correctness:

$c^d \pmod n = (m^e)^d \pmod n = m^{ed} \pmod n = m,$

because $m^{\phi(n)} \pmod n = 1$ (Fermat's little theorem)

RSA Variants and Security

Definition (Integer Factoring Problem)

Given $n = pq$, for p, q primes of the same length, find p or q .

If Factoring is easy, RSA is not secure (the secret key can be obtained from the public information).

Typical size of p and q is 1024 bits.

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Typical size of p and q is 1024 bits.

Two main ways to generate the public key:

- The public exponent e is random and coprime with $\phi(n) = (p - 1)(q - 1)$.
- The public exponent e is fixed and $p - 1$ and $q - 1$ are coprime with e .

Typical values $e = 17$ or 65537 .

Rabin PKE Scheme

KeyGen(ℓ) :

Choose random $\ell/2$ bits long primes p, q such that $p, q \equiv 3 \pmod{4}$.

$n \leftarrow pq$

Set $\mathcal{M} = \mathcal{C} = QR_n$. // the set of quadratic residues mod n

output $((n, e), (p, q))$ // factoring n must be hard!

Enc(n, m) :

output $m^2 \pmod{n}$

Dec(p, q, c) :

$m_p = c^{(p+1)/4} \pmod{p}$; $m_q = c^{(q+1)/4} \pmod{q}$

Use Chinese Remainder Theorem to compute m from (m_p, m_q) .

output m

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output m

Correctness:

Since $m \in QR_n$ then $m = x^2 \pmod{n}$

$m_p = c^{(p+1)/4} = m^{(p+1)/2} = x^{p+1} = x^2 = m \pmod{p}$

$m_q = c^{(q+1)/4} = m^{(q+1)/2} = x^{q+1} = x^2 = m \pmod{q}$

Paillier PKE Scheme

KeyGen(ℓ) :

Choose random $\ell/2$ bits long primes p, q .

$n \leftarrow pq$

Set $\mathcal{M} = \mathbb{Z}_n$ and $\mathcal{C} = \mathbb{Z}_{n^2}^\times$.

$\lambda \leftarrow \text{lcm}(p-1, q-1)$

output $(n, (n, \lambda))$ // factoring n must be hard!

Enc(n, m) :

$r \leftarrow \mathbb{Z}_n^\times$

output $(1 + mn)r^n \pmod{n^2}$

Dec(n, λ, c) : // RSA with $e = n$ gives another way to decrypt c

$c' \leftarrow c^\lambda \pmod{n^2}$

output $\frac{c'-1}{n} \lambda^{-1} \pmod{n}$

Paillier PKE Scheme

Correctness:

$$c' = c^\lambda \pmod{n^2} = (1 + mn)^\lambda r^{n\lambda} \pmod{n^2} = 1 + \lambda mn \pmod{n^2}.$$

Then, $\frac{c'-1}{n} = \frac{\lambda mn \pmod{n^2}}{n} = m\lambda \pmod{n}$

and $\frac{c'-1}{n} \lambda^{-1} \pmod{n} = m.$

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Then, $\frac{c'-1}{n} = \frac{\lambda mn \pmod{n^2}}{n} = m\lambda \pmod{n}$

and $\frac{c'-1}{n} \lambda^{-1} \pmod{n} = m.$

Another way to decrypt: Use RSA with $e = n$

$$c^d \pmod{n} = ((1 + mn)r^n)^d \pmod{n} = r^{nd} \pmod{n} = r.$$

Once r is recovered, then

$$\frac{(cr^{-n}-1) \pmod{n^2}}{n} = \frac{((1+mn)-1) \pmod{n^2}}{n} = m.$$

Paillier PKE Scheme

Definition (Integer Factoring Problem)

Given $n = pq$, for p, q primes of the same length, find p or q .

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Typical size of p and q is 1024 bits.

Variant: $1 + nm$ can be replaced by $g^m \pmod{n^2}$, for some special values of g .

Encryption is now $c = g^m r^n \pmod{n^2}$

Decryption uses the fact that $g^\lambda \pmod{n^2} = 1 + \beta n$ for some β .

Regev PKE

InstGen(ℓ) :

Select a prime q and integers n, k .

Select a Gaussian distribution $N(0, \sigma^2)$.

Set $\mathcal{M} = \{0, 1\}$.

param $\leftarrow (q, n, k, \sigma)$

KeyGen(ℓ) :

$A \leftarrow \mathbb{Z}_q^{k \times n}$; $\mathbf{s} \leftarrow \mathbb{Z}_q^{n \times 1}$; $\mathbf{x} \leftarrow \mathbb{Z}_q^{k \times 1}$;

where $x_i \leftarrow \text{round}(qN(0, \sigma^2)) \bmod q$

output $((A, \mathbf{b} = A\mathbf{s} + \mathbf{x} \bmod q), \mathbf{s})$

Enc(param, A, \mathbf{b}, m) :

$\mathbf{r} \leftarrow \{0, 1\}^{1 \times k}$

output $(\mathbf{c}_1, c_2) = (\mathbf{r}A \bmod q, \mathbf{r} \cdot \mathbf{b} + m \frac{q-1}{2} \bmod q)$

Dec(param, $\mathbf{s}, (\mathbf{c}_1, c_2)$) :

$\mu = c_2 - \mathbf{c}_1 \cdot \mathbf{s} \bmod q$

Decide m by proximity of μ to $\{0, \frac{q-1}{2}\}$

output m

Regev PKE

Correctness:

$$\begin{aligned} \mu &= c_2 - \mathbf{c}_1 \cdot \mathbf{s} \pmod{q} = \mathbf{r} \cdot \mathbf{b} + m \frac{q-1}{2} - \mathbf{rA} \cdot \mathbf{s} \pmod{q} = \\ & \mathbf{r} \cdot (\mathbf{As} + \mathbf{x}) + m \frac{q-1}{2} - \mathbf{rA} \cdot \mathbf{s} \pmod{q} = \mathbf{r} \cdot \mathbf{x} + m \frac{q-1}{2} \pmod{q}. \\ \mathbf{r} \cdot \mathbf{x} \text{ is small (Depends on } k \text{ and } \sigma) &\Rightarrow \mu \approx m \frac{q-1}{2}. \end{aligned}$$

There is a (tiny) positive probability of decryption error.

Regev PKE

Correctness:

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Typical choice for parameters:

- $n^2 < q < 2n^2$, q prime
- $k = (1 + \epsilon)(n + 1) \log q$ for some ϵ
- $\sigma = o(1/\sqrt{n} \log n)$

Security depends on some lattice theory problems (no efficient quantum attack is known!).

Summary of Constructions

- ElGamal is based on DLOG, Regev uses lattices and RSA, Rabin and Paillier are based on Factoring
- ElGamal can be based on elliptic curves for shorter key sizes
- All except Regev are broken with a quantum computer
- RSA and Rabin are deterministic, while the others are probabilistic
- The ratio of ciphertext/plaintext sizes is 1 for RSA and Rabin, 2 for ElGamal and Paillier, and it is huge for Regev

Homomorphic Encryption

Let $\Pi = (\text{KeyGen}, \text{Enc}, \text{Dec})$ be a PKE scheme with message space \mathcal{M} and ciphertext space \mathcal{C} . Let \otimes be a binary operation defined on \mathcal{M} .

Definition (Weakly Homomorphic Encryption)

Π is weakly homomorphic with respect to \otimes if there exists an efficient algorithm $\text{HomEval}(\cdot)$ such that for all properly generated (pk, sk) and all messages $m_1, m_2 \in \mathcal{M}$

$$\text{Dec}(sk, \text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))) = m_1 \otimes m_2$$

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$$\text{Dec}(sk, \text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))) = m_1 \otimes m_2$$

It means computing on encrypted data without decrypting it!

Examples

Typically, there exist efficient group operations, say \otimes in \mathcal{M} and \odot on \mathcal{C} , such that

$$\text{Dec}(sk, c_1 \odot c_2) = \text{Dec}(sk, c_1) \otimes \text{Dec}(sk, c_2)$$

and one can just define $\text{HomEval}(pk, c_1, c_2) = c_1 \odot c_2$ to obtain a weak homomorphic PKE.

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Example: RSA is homomorphic w.r.t. multiplication in \mathbb{Z}_n^\times

$$\begin{aligned} \text{Dec}(n, d, c_1 c_2 \bmod n) &= (c_1 c_2)^d \bmod n = c_1^d c_2^d \bmod n = \\ &= \text{Dec}(n, d, c_1) \text{Dec}(n, d, c_2) \bmod n \end{aligned}$$

or

$$\begin{aligned} \text{Enc}(n, e, m_1) \text{Enc}(n, e, m_2) \bmod n &= m_1^e m_2^e \bmod n = \\ &= (m_1 m_2)^e \bmod n = \text{Enc}(n, e, m_1 m_2 \bmod n) \end{aligned}$$

Strongly Homomorphic Encryption

Weak homomorphism can leak some information about

- whether a ciphertext c is computed via HomEval or directly with Enc,
- whether certain ciphertext c_1 was used in a call to HomEval.

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Definition (Strongly Homomorphic Encryption)

Π is strongly homomorphic if in addition, for any valid (pk, sk) , m_1 and m_2 , the random variables $\text{Enc}(pk, m_1 \otimes m_2)$ and $\text{HomEval}(pk, \text{Enc}(pk, m_1), \text{Enc}(pk, m_2))$ are independent and identically distributed.

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Lemma

Any weakly homomorphic deterministic PKE is also strongly.

Ciphertext Rerandomization

In a strongly homomorphic probabilistic PKE, $\text{HomEval}(\cdot)$ needs to remove any correlation between the randomness in $\text{Enc}(pk, m_1)$, $\text{Enc}(pk, m_2)$ and $\text{Enc}(pk, m_1 \otimes m_2)$.

Main idea: Define $\text{ReRand}(pk, \cdot)$, so that it transforms any ciphertext c into another one c' such that

- $\text{Dec}(sk, c) = \text{Dec}(sk, c')$,
- c and c' have independent randomness,
- c' has the same probability distribution as $\text{Enc}(pk, m)$.

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- c and c' have independent randomness,
- c' has the same probability distribution as $\text{Enc}(pk, m)$.

E.g., $\text{ReRand}(pk, c) = \text{WeakHomEval}(pk, c, \text{Enc}(pk, m_0))$, where m_0 is the neutral element (0 for addition, 1 for multiplication)

Then, we simply define:

$\text{HomEval}(pk, c_1, c_2) = \text{ReRand}(pk, \text{WeakHomEval}(pk, c_1, c_2))$.

Examples

ElGamal is strongly homomorphic in G :

$\text{Enc}(y, m_1) \cdot \text{Enc}(y, m_2) = (g^{r_1}, y^{r_1} m_1) \cdot (g^{r_2}, y^{r_2} m_2) = (g^{r_1+r_2}, y^{r_1+r_2} m_1 m_2)$ is an encryption of $m_1 m_2$.

$\text{HomEval}(y, c_1, c_2) = c_1 \cdot c_2 \cdot (g^{r_3}, y^{r_3})$ for a random $r_3 \in \mathbb{Z}_q$.

It results in $(g^r, y^r m_1 m_2)$ for $r = r_1 + r_2 + r_3$, which is independent of r_1 and r_2 .

Examples

Paillier is (additive) strongly homomorphic in \mathbb{Z}_n :

$\text{Enc}(n, m_1)\text{Enc}(n, m_2) \bmod n^2 = (1 + m_1 n)r_1^n(1 + m_2 n)r_2^n$
 $\bmod n^2 = (1 + (m_1 + m_2)n)(r_1 r_2)^n \bmod n^2$ is an encryption of
 $m_1 + m_2 \bmod n$.

$\text{HomEval}(n, c_1, c_2) = c_1 c_2 r_3^n \bmod n^2$ for a random $r_3 \in \mathbb{Z}_n^\times$.

It results in $(1 + (m_1 + m_2)n)r^n \bmod n^2$ for $r = r_1 r_2 r_3 \bmod n$,
which is independent of r_1 and r_2 .

Homomorphic Encryption Summary

- RSA is homomorphic w.r.t. multiplication mod n , for $n = pq$.
- ElGamal is homomorphic w.r.t. the operation in the group G .
- Hashed ElGamal has no homomorphic property due to the hash function.
- Paillier is homomorphic w.r.t. addition mod n , for $n = pq$.
- Regev is weakly homomorphic w.r.t. addition mod 2 (XOR operation).

A client application would be interested in common logical or arithmetic operations on bitstrings (none of the above are!).

Applications

If $n = pq$ is large enough, Paillier is quite homomorphic for normal addition of positive integers \Rightarrow Lots of applications!

It can also be done with ElGamal and exponentiated messages $m = g^x$. Then, $g^{x_1} g^{x_2} = g^{x_1+x_2}$, but decryption (i.e. recovering x) can only be done if x is small.

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Example: (very simplistic) electronic voting

- Encrypted ballot: $c_i = \text{Enc}(pk, \textit{ballot})$ where $\textit{ballot} \in \{0, 1\}$
- Aggregate ballot: $c \leftarrow \text{HomEval}(pk, c_1, \dots, c_n)$
- Tally computation: $\textit{tally} = \text{Dec}(sk, c)$, where $\textit{tally} = \sum_i \textit{ballot} = \#\{\textit{ballot} = 1\}$

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Many verifications should be added for a realistic voting scheme (detection of corrupted ballots, multiple voting, corrupted tally. . .) while preserving voter's privacy.

Fully Homomorphic Encryption

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Arithmetic circuits over the ring \mathbb{Z}_n include boolean circuits, because if one encodes logical '0' and '1' as arithmetic '0' and '1', then

- x_1 **AND** $x_2 = x_1 x_2$
- x_1 **OR** $x_2 = x_1 + x_2 - x_1 x_2$
- x_1 **XOR** $x_2 = x_1 + x_2 - 2x_1 x_2$
- **NOT** $x = 1 - x$

Data Protection

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MCYBERS, UPC, Fall 2025

END