



# Outline

- 1 Defining Computational Security
- 2 Security Models for Public Key Encryption
- 3 Security Models for Digital Signatures
- 4 Security Assumptions and Results

# Bounding the Adversary's Capabilities

## Perfect Security vs. Computational Security

- **Perfect Security:** The adversary is unbounded, but only a few positive results can be obtained. (E.g., perfect one-time encryption.)



# Bounding the Adversary's Capabilities

## Perfect Security vs. Computational Security

- **Perfect Security:** The adversary is unbounded, but only a few positive results can be obtained. (E.g., **perfect one-time encryption.**)
- **Computational Security:** The adversary is bounded in
  - running time
  - memory usage
  - success probability

Security is based on the assumption that some computational problem (family) is hard.

We need to define:

- A notion of problem hardness. (**Asymptotic analysis.**)
- A computation model. (**Turing machine.**)



# Turing Machine

A model for mechanic computation, consisting of

- A finite automaton
  - state information  $s \in S \cup \{\text{init}, \text{halt}\}$
  - transition function  $f$
- A storage device (“tape”)

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Computation step:  $(s, wr, mv) \leftarrow f(s, c)$

$wr \in \{\text{write}_0, \text{write}_1, \text{erase}\}$

$mv \in \{\text{move\_left}, \text{move\_right}\}$

$c =$  contents of the current cell

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Input/Output of the Turing Machine is the content of the tape in the `init/halt` state

# Turing Machine

state

$s$

transition func.

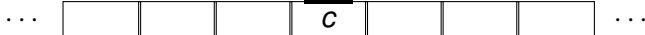
$(s, c)$

$f$

$(s^+, wr, mv) = f(s, c)$

read/write head

tape



# Turing Machine

state

The diagram illustrates the components of a Turing Machine. At the top, a box labeled 's' represents the current state. Below it is a box labeled 'f' representing the transition function. At the bottom, a horizontal tape is shown with a black square representing the read/write head. The current cell of the tape is labeled 'c'. Arrows indicate the flow of information: from the state 's' to the transition function 'f', from the transition function 'f' back to the state 's', from the transition function 'f' to the read/write head, and from the tape cell 'c' to the read/write head. The transition function 'f' takes an input pair  $(s, c)$  and produces an output triple  $(s^+, wr, mv) = f(s, c)$ . The read/write head has two arrows pointing left and right, indicating its movement along the tape.

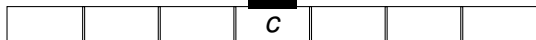
transition func.

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read/write head

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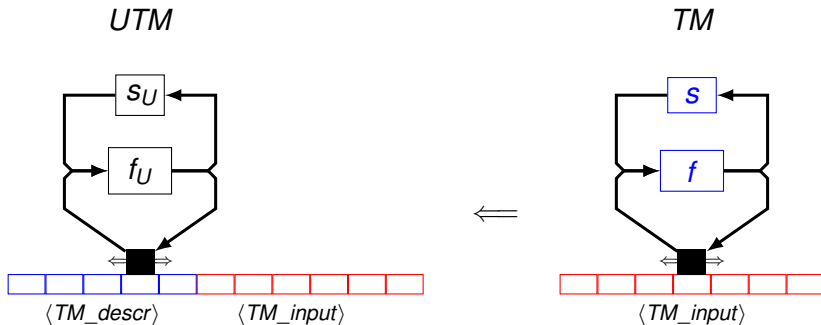
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...

 $s \in \mathcal{S} \cup \{\text{init}, \text{halt}\}$  $wr \in \{\text{write}_0, \text{write}_1, \text{erase}\}$  $mv \in \{\text{move\_left}, \text{move\_right}\}$

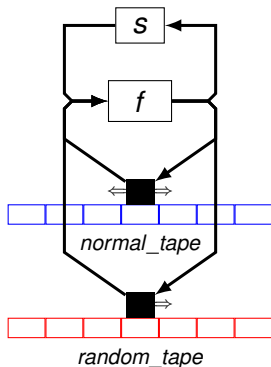
# Universal Turing Machine



It can simulate any known (classical) computing device with reasonable efficiency

# Probabilistic Turing Machine

*PTM*



A Turing Machine that take random decisions, from an additional input tape.

Computation step:

$$(s, wr, mv, mvr) \leftarrow f(s, c, r)$$

$wr \in \{\text{write}_0, \text{write}_1, \text{erase}\}$

$mv \in \{\text{move\_left}, \text{move\_right}\}$

$mvr \in \{\text{keep}, \text{move\_right}\}$

$c$  = contents of the current normal cell

$r$  = contents of the current random cell

# Algorithmic Complexity

A running time limitation (number of steps of the Probabilistic Turing Machine) implies similar space and randomness limitations.

Every read/write operations takes one step.

Reading a random bit takes one step.

The running time typically depends on the size of the input (description of the problem to be solved).

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**Uniform approach:** A single Turing Machine tries to solve problems of all sizes.

**Asymptotic analysis:** We study the problem complexity by the type of growth of the running time as a function of the input size.

# Hardness Notions

**Easy computation:** For a problem family  $\mathcal{P}$ , there exists a Turing Machine TM such that

- The running time is polynomial on the size of  $P \in \mathcal{P}$ .
- It always outputs a correct solution of any problem instance  $P \in \mathcal{P}$

$$\exists T(\cdot) \in \mathbf{poly} \quad \forall P \in \mathcal{P}$$

$$\text{time(TM}, P) \leq T(|P|) \quad \text{and} \quad \text{valid}(P, TM(P)) = 1$$

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**poly:** The set of positive polynomials.

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A model for normal operations performed by honest parties in a protocol.

---

**poly:** The set of positive polynomials.

# Hardness Notions

**Feasible computation:** There exists a Probabilistic Turing Machine TM such that

- The running time is polynomial on the size of  $P \in \mathcal{P}$ .
- The probability that it gives a correct solution for any problem instance  $P \in \mathcal{P}$  is not *negligible*.

$\exists T(\cdot), Q(\cdot) \in \mathbf{poly} \quad \forall P \in \mathcal{P}$

$\text{time}(\text{TM}, P) \leq T(|P|)$  and  $\Pr(\text{valid}(P, \text{TM}(P)) = 1) \geq 1/Q(|P|)$

---

**negl:** The set of functions  $f(\cdot)$  such that for any  $Q(\cdot) \in \mathbf{poly}$  there exists  $n_0$  s.t.  $f(n) < 1/Q(n)$  for all  $n > n_0$ .

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$\exists T(\cdot), Q(\cdot) \in \mathbf{poly} \quad \forall P \in \mathcal{P}$

$\text{time}(\text{TM}, P) \leq T(|P|)$  and  $\Pr(\text{valid}(P, \text{TM}(P)) = 1) \geq 1/Q(|P|)$

A model for a successful attack against a protocol.

---

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# Hardness Notions

## Summary:

- **Easy:** There exists a TM that solves all problem instances in polynomial time.
- **Feasible:** There exists a PPTM that solves any problem instance with a non-negligible probability.
- **Infeasible:** All PPTM can only solve problem instances with a negligible probability.

In cryptography, the problem  $P$  is chosen at random among all problems with a specific size (e.g., the description of the problem instance contains a randomly generated public key).

# Security as a Game

Security is modelled as a Game (random experiment) between a Challenger  $C$  and an Adversary  $A$ :

- $C$  chooses a random problem instance of a given size
- $A$  tries to find a solution in polynomial time
- $A$  succeeds only with a negligible probability

Here, the success probability is computed respect to

- The internal coin tosses of  $A$
- The random choice of  $C$

In general, the game can have more interaction steps.



# One-Time Security for Symmetric Encryption

The adversary tries to decrypt a random ciphertext from scratch.

Challenger

$\leftarrow (\ell, \text{KeyGen}, \text{Enc}, \text{Dec}) \rightarrow$

Adversary

$k_* \leftarrow \text{KeyGen}(\ell)$

$m_* \leftarrow \mathcal{M}$

$c_* = \text{Enc}(k_*, m_*)$ ; **send**( $c_*$ )

**receive**( $c_*$ )

???

**receive**( $m'$ )

**send**( $m'$ )

**accept if**  $m' = m_*$

It is too simplistic: the adversary does not know any message / ciphertext pair for the same key. (It is a one-time model.)

# One-Time Security for Symmetric Encryption

The same, written in pseudocode style (we don't know what the adversary is really doing):

**Experiment**  $\text{Exp-SE-OT}(\Pi, \mathcal{A}, \ell)$  :

$k_* \leftarrow \text{KeyGen}(\ell)$

$m_* \leftarrow \mathcal{M}$

$c_* \leftarrow \text{Enc}(k_*, m_*)$

$m' \leftarrow \mathcal{A}(c_*)$

**if**  $m' = m_*$  **output** 1;     //  $\mathcal{A}$  wins

**else output** 0;

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**if**  $m' = m_*$  **output** 1;     //  $\mathcal{A}$  wins

**else output** 0;

## Definition (SE-OT)

The symmetric encryption scheme  $\Pi$  is SE-OT secure if for all PPTM  $\mathcal{A}$

$$\Pr[\text{Exp-SE-OT}(\Pi, \mathcal{A}, \ell) = 1] \in \mathbf{negl}(\ell)$$

# One-Way Security for Public Key Encryption

Now, the adversary also receives the public key:  
 It can encrypt messages of its choice with the same target key  
 (CPA model)

Challenger

$\leftarrow (\ell, \text{KeyGen}, \text{Enc}, \text{Dec}) \rightarrow$

Adversary

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$m_* \leftarrow \mathcal{M}$

$c_* = \text{Enc}(pk_*, m_*); \text{ send}(pk_*, c_*)$

**receive**( $m'$ )

**accept if**  $m' = m_*$

**receive**( $pk_*, c_*$ )

???

**send**( $m'$ )

# One-Way Security for Public Key Encryption

**Experiment**  $\text{Exp-PKE-OW-CPA}(\Pi, \mathcal{A}, \ell)$  :

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$m_* \leftarrow \mathcal{M}$

$c_* \leftarrow \text{Enc}(pk_*, m_*)$

$m' \leftarrow \mathcal{A}(pk_*, c_*)$

**if**  $m' = m_*$  **output** 1;     //  $\mathcal{A}$  wins

**else output** 0;

## Definition (PKE-OW-CPA)

The public key encryption scheme  $\Pi$  is PKE-OW-CPA secure if for all PPTM  $\mathcal{A}$

$$\Pr[\text{Exp-PKE-OW-CPA}(\Pi, \mathcal{A}, \ell) = 1] \in \mathbf{negl}(\ell)$$

# One-Way Security for Public Key Encryption

Most proposed PKE schemes are conjectured to be OW-CPA secure, with a proper choice of the parameters.

E.g., ElGamal, RSA, Rabin, Paillier.

Regev's PKE is not OW-CPA secure because it encrypts a single bit (thus the adversary has easily a success probability of  $1/2$ ).

There is a known security proof showing that Rabin is PKE-OW-CPA secure if and only if factoring an RSA modulus (with  $p, q = 3 \pmod{4}$ ) is hard.

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PKE-OW-CPA security is still unrealistic when the adversary has some a priori knowledge about the target message  $m_*$ .

# Semantic Security

The adversary chooses some leakage function  $f$  for  $m_*$ :

Challenger

$\leftarrow (\ell, \text{KeyGen}, \text{Enc}, \text{Dec}) \rightarrow$

Adversary

$b \leftarrow \{0, 1\}$

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

**send** $(pk_*)$

**receive** $(f(\cdot))$

$m_0, m_1 \leftarrow \mathcal{M}; h = f(m_0)$

$c_* = \text{Enc}(pk_*, m_b); \text{send}(c_*, h)$

**receive** $(b')$

**accept if**  $b' = b$

**receive** $(pk_*)$

???

**send** $(f(\cdot))$

(keep the internal state  $st$ )

**receive** $(c_*, h)$

???

**send** $(b')$

This is a two-stage game (the Adversary is split in two stages).

# Indistinguishability of Ciphertexts

Semantic Security game is shown to be equivalent to:

Challenger

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Adversary

$b \leftarrow \{0, 1\}$

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

**send** $(pk_*)$

**receive** $(m_0, m_1)$

$c_* = \text{Enc}(pk_*, m_b)$

**send** $(c_*)$

**receive** $(b')$

**accept if**  $b' = b$

**receive** $(pk_*)$

???

**send** $(m_0, m_1)$

(keep the internal state  $st$ )

**receive** $(c_*)$

???

**send** $(b')$

$m_0, m_1$  must be valid messages. (Otherwise, the Challenger rejects).

# Indistinguishability of Ciphertexts

**Experiment**  $\text{Exp-PKE-IND-CPA}(\Pi, \mathcal{A}_1, \mathcal{A}_2, \ell)$  :

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$(m_0, m_1, st) \leftarrow \mathcal{A}_1(pk_*)$  //  $st$  is the internal state passing from  $\mathcal{A}_1$  to  $\mathcal{A}_2$

$b \leftarrow \{0, 1\}$

$c_* \leftarrow \text{Enc}(pk_*, m_b)$

$b' \leftarrow \mathcal{A}_2(st, c_*)$

**if**  $b' = b$  **output** 1; //  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$  wins

**else output** 0;

## Definition (PKE-IND-CPA)

The public key encryption scheme  $\Pi$  is PKE-IND-CPA secure if for all PPTM  $\mathcal{A}$

$$|\Pr[\text{Exp-PKE-IND-CPA}(\Pi, \mathcal{A}, \ell) = 1] - 1/2| \in \mathbf{negl}(\ell)$$

$1/2$  is the success probability of a dummy adversary!

# IND-CPA Security

Some well-known PKE schemes are conjectured to be IND-CPA secure, with a proper choice of the parameters.

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## Lemma

*No deterministic PKE can achieve IND-CPA security.*

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RSA and Rabin need some randomization (e.g., a randomized message padding scheme).

# One-Way Functions

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## Definition

A function family  $\mathcal{F} = \{\mathcal{F}_\ell\}_{\ell \in \mathbb{Z}^+}$ ,  $\mathcal{F}_\ell = \{f_k : \mathcal{X}_k \rightarrow \mathcal{Y}_k\}_{k \in \mathcal{K}_\ell}$  is **one-way** if it is efficiently computable, but for all PPTM  $\mathcal{A}$

$$\Pr[\mathcal{A}(1^\ell, k, y) \in f_k^{-1}(y) : k \leftarrow \mathcal{K}_\ell; x \leftarrow \mathcal{X}_k; y \leftarrow f_k(x)] \in \mathbf{negl}(\ell)$$

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- ‘Efficiently computable’ means that there is a PPTM Eval such that  $\text{Eval}(k, x) = f_k(x)$
- If  $f_k$  is one-way then  $f'_k$  defined by  $f'_k(x, r) = (f_k(x), r)$  is also one-way
- The sets  $\mathcal{X}_k$  and  $\mathcal{Y}_k$  must be of size superpolynomial in  $\ell$

# Injective Trapdoor One-Way Functions

PKE also requires the existence of a **trapdoor** which knowledge renders the decryption function easy to compute.

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An injective one-way function family  $\mathcal{F} = \{\mathcal{F}_\ell\}_{\ell \in \mathbb{Z}^+}$ ,  $\mathcal{F}_\ell = \{f_k : \mathcal{X}_k \rightarrow \mathcal{Y}_k\}_{k \in \mathcal{K}_\ell}$  is called **trapdoor one-way** if there exists a family of trapdoors  $\mathcal{T} = \{\mathcal{T}_\ell\}_{\ell \in \mathbb{Z}^+}$ , and two PPTM Sample and Inv such that

$$\Pr[\text{Inv}(1^\ell, t, f_k(x)) = x : (k, t) \leftarrow \text{Sample}(1^\ell); x \leftarrow \mathcal{X}_k] = 1$$

and  $(k, t) \leftarrow \text{Sample}(1^\ell)$  samples the uniform distribution in  $\mathcal{K}_\ell$ , and  $t \in \mathcal{T}_\ell$

# PKE From Injective TOW Functions

Let  $\mathcal{F}$  be an injective trapdoor one-way function family.

KeyGen( $\ell$ ) :

$(k, t) \leftarrow \text{Sample}(\ell)$ ;

**output**  $(k, t)$ ;

Enc( $k, m$ ) :

**output** Eval( $k, m$ );

Dec( $t, c$ ) :

**output** Inv( $t, c$ );

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It is PKE-OW-CPA secure but not PKE-IND-CPA secure  
(because the encryption function is deterministic)

# Hardcore Predicates of a One-Way Function

Let  $\mathcal{F}$  be an injective one-way function family between the set families  $\mathcal{X}$  and  $\mathcal{Y}$ . A family of predicates  $\mathcal{H}$  (functions taking binary values) on  $\mathcal{X}$  is hardcore for  $\mathcal{F}$  if computing  $h_k(x)$  from  $f_k(x)$  is as hard as computing  $x$  from  $f_k(x)$ .

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Examples:

- For an RSA public key  $(n = pq, e)$ , computing  $LSB(x)$  from  $x^e \pmod n$  is as hard as computing  $x$  from  $x^e \pmod n$
- **Goldreich-Levin predicate:**

$$h(x, r) = x_1 r_1 + \dots + x_n r_n \pmod 2$$

is a hardcore predicate for  $f'_k(x, r) = (f_k(x), r)$

# PKE From Hardcore Predicates

**Trapdoor One-Way Permutation (TOWP) Family:** A trapdoor one-way family of bijections  $f_k : \mathcal{X}_k \rightarrow \mathcal{X}_k$   
 $\mathcal{H}$  hardcore predicate family for  $\mathcal{F}$

KeyGen( $\ell$ ) :

$(k, t) \leftarrow \text{Sample}(\kappa(\ell));$

**output**  $(k, t);$

Enc( $k, m$ ) :

$(m_1, \dots, m_\ell) = m;$

$r \leftarrow \mathcal{X}_k;$

$c_i \leftarrow m_i \oplus h_k(f_k^{i-1}(r)); \quad i = 1, \dots, \ell$

**output**  $(c_1, \dots, c_\ell, f_k^\ell(r));$

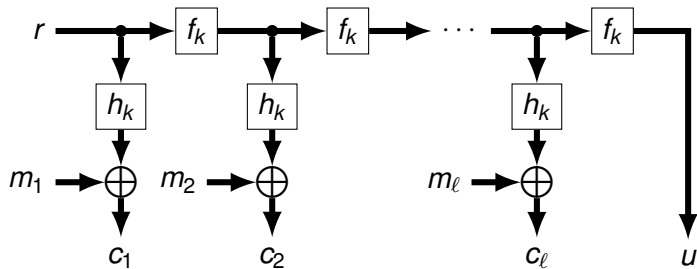
Dec( $t, c$ ) :

$(c_1, \dots, c_\ell, u) = c;$

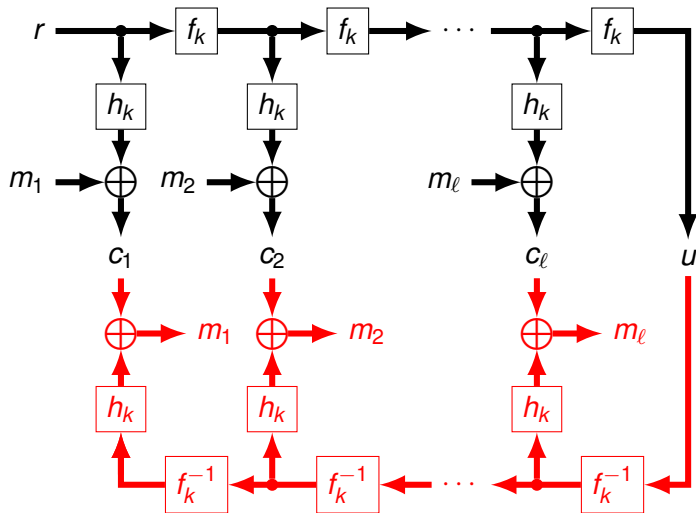
$m_i \leftarrow c_i \oplus h_k(f_k^{-(\ell-i+1)}(u)); \quad i = 1, \dots, \ell$

**output**  $m = (m_1, \dots, m_\ell);$

# PKE From Hardcore Predicates



# PKE From Hardcore Predicates



# Beyond IND-CPA Security

Some realistic attacks fall outside the IND-CPA model.

The adversary has limited extra access to:

- An oracle that tells whether a (possibly manipulated) ciphertext is valid or not.
- An oracle that decrypts a (possibly manipulated) ciphertext.
- An oracle that decrypts a ciphertext related to the target one  $c_*$ .

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- An oracle that decrypts a ciphertext related to the target one  $c_*$ .

A maximal notion of security is defined: IND-CCA (for Chosen Ciphertext Attack).

The adversary can ask for decryptions of any possible ciphertext except for  $c_*$ .

# IND-CCA Security

Challenger

 $\leftarrow (\ell, \text{KeyGen}, \text{Enc}, \text{Dec}) \rightarrow$ 

Adversary

 $(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$ 
**send** $(pk_*)$ 
**receive** $(pk_*)$ 

(many times)

**receive** $(c_i)$ 
 $m_i = \text{Dec}(sk_*, c_i)$ ; **send** $(m_i)$ 

???

**send** $(c_i)$  // oracle call

**receive** $(m_i)$ 
**receive** $(m_0, m_1)$ 
 $b \leftarrow \{0, 1\}$ ;  $c_* = \text{Enc}(pk_*, m_b)$ ; **send** $(c_*)$ 

???

**send** $(m_0, m_1)$ 
**receive** $(c_*)$ 

(many times)

**receive** $(c_i)$ 
**if**  $c_i \neq c_*$ ;  $m_i = \text{Dec}(sk_*, c_i)$ ; **send** $(m_i)$ 
**else abort**

???

**send** $(c_i)$  // oracle call

**receive** $(m_i)$ 
**receive** $(b')$ 
**accept if**  $b' = b$ 

???

**send** $(b')$

# IND-CCA Security

**Experiment**  $\text{Exp-PKE-IND-CCA}(\Pi, \mathcal{A}_1, \mathcal{A}_2, \ell)$  :

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$(m_0, m_1, st) \leftarrow \mathcal{A}_1^{\mathcal{O}_1(\cdot)}(pk_*)$

$b \leftarrow \{0, 1\}$

$c_* \leftarrow \text{Enc}(pk_*, m_b)$

$b' \leftarrow \mathcal{A}_2^{\mathcal{O}_2(\cdot)}(st, c_*)$

**if**  $b' = b$  **output** 1;     //  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$  wins

**else output** 0;

**Oracle**  $\mathcal{O}_1(c)$      // Decryption oracle

**return**  $\text{Dec}(sk_*, c)$

**Oracle**  $\mathcal{O}_2(c)$      // Restricted decryption oracle

**if**  $c \neq c_*$  **return**  $\text{Dec}(sk_*, c)$  **else abort**

## Definition (PKE-IND-CCA)

The PKE  $\Pi$  is PKE-IND-CCA secure if for all PPTM  $\mathcal{A}$

$$|\Pr[\text{Exp-PKE-IND-CCA}(\Pi, \mathcal{A}, \ell) = 1] - 1/2| \in \text{negl}(\ell)$$

# Known CCA Attacks

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- **Paillier**: The same attack also applies.
- **Hashed ElGamal**: XORing the second component of  $c_*$  with any mask  $z$  makes the oracle answer  $m_b \oplus z$ .
- **Regev**: Simply adding  $(q - 1)/2$  to the second component of  $c_*$  makes the oracle answer  $1 - m_b$ .

# Known CCA Attacks

With similar attacks, it is shown that:

Lemma

*Any homomorphic PKE is IND-CCA insecure.*

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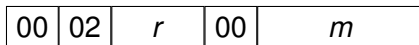
New methods are necessary to upgrade existing PKE to the IND-CCA security level.

- Randomized paddings, or Fujisaki-Okamoto Transforms (in the Random Oracle Model)
- New cryptographic tools (Hash Proof Systems, Canetti-Halevi-Katz Transform, . . .)

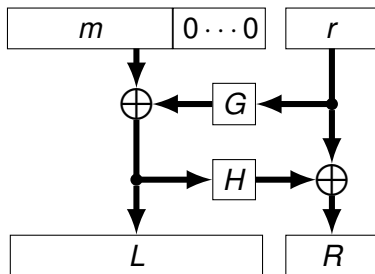
# E.g., Fixing RSA PKE

- Plain RSA PKE is conjectured to be OW-CPA secure, but it is neither IND-CPA nor OW-CCA secure.
- A simple randomized message padding can give IND-CPA security, but not OW-CCA.
- Bleichenbacher's attack against PKCS#1 v1.5 shows a practical CCA attack against a padded version of RSA.
- RSA-OAEP (PKCS#1 v2) fixes the attack and provides IND-CCA security for RSA (in the Random Oracle Model).

# PKCS#1 v1.5 vs. v2.0 Message Encodings



PKCS#1 v1.5



PKCS#1 v2.0

# Outline

- 1 Defining Computational Security
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- 3 Security Models for Digital Signatures**
- 4 Security Assumptions and Results

# Universal Unforgeability (UF)

The basic security notion for signatures: forge a valid signature for any given message only from the public key:

Challenger

$\leftarrow (\ell, \text{KeyGen}, \text{Sig}, \text{Ver}) \rightarrow$

Adversary

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$m_* \leftarrow \mathcal{M}$

**send** $(pk_*, m_*)$

**receive** $(s_*)$

**accept if**  $\text{Ver}(pk_*, m_*, s_*) = 1$

**receive** $(pk_*, m_*)$

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Actually, the random selection of  $m_*$  is not well defined!.

# UF-RMA Security

In UF-RMA security, the adversary can ask for valid signatures on random messages.

**Experiment**  $\text{Exp-UF-RMA}(\Sigma, \mathcal{A}, \ell)$  :

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$s_* \leftarrow \mathcal{A}^{\mathcal{O}()}(pk_*, m_*)$

**if**  $\text{Ver}(pk_*, m_*, s_*) = 1$  **output** 1;     //  $\mathcal{A}$  wins

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The signature scheme  $\Sigma$  is UF-RMA secure if for all PPTM  $\mathcal{A}$

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*Again, the random selection of  $m_*$  and  $m$  are not well defined!*

**Improvement:** Make the adversary choose the target message to be signed.

# Existential Unforgeability (EF)

**Experiment**  $\text{Exp-EF-RMA}(\Sigma, \mathcal{A}, \ell)$  :

$(pk_*, sk_*) \leftarrow \text{KeyGen}(\ell)$

$L = \emptyset$

$(m_*, s_*) \leftarrow \mathcal{A}^{O()}(pk_*)$

**if**  $m_* \notin L$  **and**  $\text{Ver}(pk_*, m_*, s_*) = 1$  **output** 1; //  $\mathcal{A}$  wins

**else output** 0;

**Oracle**  $O()$  // Signing a random message oracle

$m \leftarrow \mathcal{M}$

$L = L \cup \{m\}$

**return**  $(m, \text{Sig}(sk_*, m))$

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We need to maintain the list of messages signed by the oracle to exclude trivial attacks.

# Beyond RMA Security

Plain RSA and ElGamal signatures are EF-RMA insecure, while RSA-FDH, Pointcheval-Stern, DSA and ECDSA are EF-RMA secure.

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In a practical setting, there exist ways to inject some messages to be signed by a honest signer, and this is a type of attack that is outside the RMA model.

**Improvement:** Allow the adversary choose the messages to be signed by the oracle.

# EF-CMA Security

**Experiment**  $\text{Exp-EF-CMA}(\Sigma, \mathcal{A}, \ell)$  :

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# EF-CMA Security

EF-CMA is nowadays considered the standard security level for a digital signature scheme.

RSA-FDH, Pointcheval-Stern, DSA and ECDSA achieve EF-CMA security in the Random Oracle Model.

There are other signature schemes (e.g., Cramer-Shoup) that also achieve EF-CMA security without random oracles.

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# Assumptions Related to Integer Factorization

- **IF:** For random  $\ell$ -bit long primes  $p, q$ , **given**  $n = pq$ , **compute**  $p$  or  $q$ .

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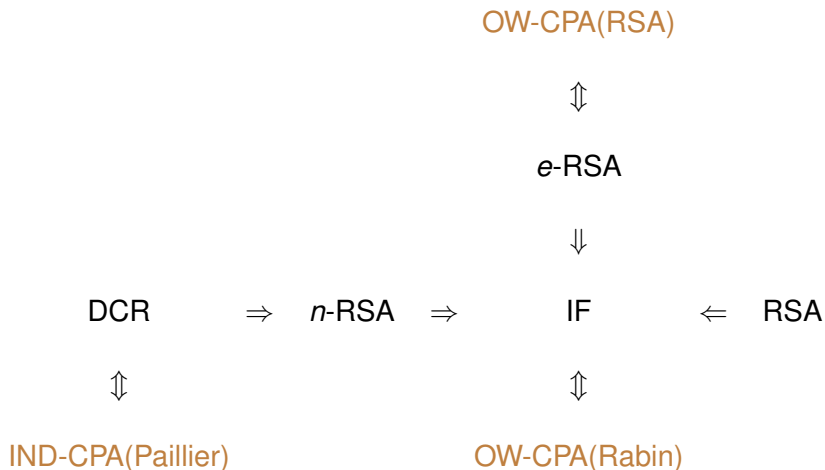
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- **e-RSA:** For random  $\ell$ -bit long primes  $p, q$  such that  $e$  does not divide  $p - 1$  or  $q - 1$  and a random  $m \in \mathbb{Z}_n$ , **given**  $n$  and  $m^e \bmod n$ , **compute**  $m$ .





# Known Hardness Implications



# Assumptions Related to Discrete Logarithm

For a  $\ell$ -bit long prime  $q$ , a cyclic group  $G$  of order  $q$  and a generator  $g$ :

- **DLOG:** For random  $x \in \mathbb{Z}_q$ , **given**  $g^x$ , **compute**  $x$ .



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- **CDH:** For random  $x, y \in \mathbb{Z}_q$ , **given**  $g^x$  and  $g^y$ , **compute**  $g^{xy}$ .
- **DDH:** For random  $x, y, z \in \mathbb{Z}_q$ , **tell apart** the two probability distributions  $(g^x, g^y, g^z)$  and  $(g^x, g^y, g^{xy})$ .



# Known Hardness Implications

DDH  $\Rightarrow$  CDH  $\Rightarrow$  DLOG

$\Updownarrow$

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IND-CPA(EIGamal)  $\Rightarrow$  OW-CPA(EIGamal)

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In some special groups (e.g., elliptic curves with efficient pairing maps) DDH is easy while CDH is still conjectured to be hard.

Secure variants of ElGamal encryption and new encryption and signature schemes with new features have been proposed in these special groups (pairing based cryptography).

# Further Assumptions

There exist other settings that offer well-known problems also conjectured hard, that can be used to build cryptographic protocols:

- Lattices
- Coding Theory
- Multivariate Polynomials
- Isogenies
- Non-abelian Groups

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Some of the problems could remain hard even in the presence of quantum computers.

# Cryptology

Jorge L. Villar

FME, UPC, Fall 2024

**END**