IOP Publishing Nonlinearity

Nonlinearity **21** (2008) 1759–1811 doi:10.1088/0951-7715/21/8/005

Kolmogorov–Arnold–Moser aspects of the periodic Hamiltonian Hopf bifurcation

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Received 25 October 2007, in final form 22 May 2008
Published 26 June 2008
Online at stacks.iop.org/Non/21/1759

Recommended by A Chenciner

Abstract

In this work we consider a $1:-1$ non-semi-simple resonant periodic orbit of a three degrees of freedom real analytic Hamiltonian system. From the formal analysis of the normal form, we prove the branching off of a two-parameter family of two-dimensional invariant tori of the normalized system, whose normal behaviour depends intrinsically on the coefficients of its low-order terms. Thus, only elliptic or elliptic together with parabolic and hyperbolic tori may detach from the resonant periodic orbit. Both patterns are mentioned in the literature as the direct and inverse, respectively, periodic Hopf bifurcation. In this paper we focus on the direct case, which has many applications in several fields of science. Our target is to prove, in the framework of Kolmogorov–Arnold–Moser (KAM) theory, the persistence of most of the (normally) elliptic tori of the normal form, when the whole Hamiltonian is taken into account, and to give a very precise characterization of the parameters labelling them, which can be selected with a very clear dynamical meaning. Furthermore, we give sharp quantitative estimates on the 'density' of surviving tori, when the distance to the resonant periodic orbit goes to zero, and show that the four-dimensional invariant Cantor manifold holding them admits a Whitney-C^∞ extension. Due to the strong degeneracy of the problem, some standard KAM methods for elliptic low-dimensional tori of Hamiltonian systems do not apply directly, so one needs to properly suit these techniques to the context.

Mathematics Subject Classification: 37J20, 37J40

1. Introduction

This paper is related to the existence of quasiperiodic solutions linked to a Hopf bifurcation scenario in the Hamiltonian context. In its simpler formulation, we shall consider a real analytic three degrees of freedom Hamiltonian system with a one-parameter family of periodic orbits undergoing a $1:-1$ resonance for some value of the parameter. By $1:-1$ resonance we mean that, for the corresponding *resonant* or *critical* periodic orbit, a pairwise collision of its characteristic non-trivial multipliers (i.e. those different from 1) takes place at two conjugate points on the unit circle. When varying the parameter it turns out that, generically, prior to the collision the non-trivial multipliers are different and lie on the unit circle (by conjugate pairs) and after the collision, they move out, by reciprocal pairs, into the complex plane forming a complex quadruplet so the periodic orbits of the family become unstable.

This mechanism of destabilization is often referred to in the literature as *complex instability* (see [24]) and has also been studied for families of four-dimensional symplectic maps, where an elliptic fixed point evolves to a complex saddle as the parameter of the family moves (see [16, 45]). Under general conditions, the branching off of two-dimensional quasiperiodic solutions (respectively, invariant curves for mappings) from the resonant periodic orbit (respectively, the fixed point) has been described both numerically (in [15, 25, 38, 41, 42, 44]) and analytically (in [7, 23, 39, 43]).

This phenomenon has some straightforward applications, for instance, in celestial mechanics. Indeed, let us consider the so-called *vertical family* of periodic orbits of the (Lagrange) equilibrium point L_4 in the (spatial) restricted three body problem, that is, the Lyapunov family associated with the vertical oscillations of L_4. It turns out that, for values of the mass parameter greater than Routh's value, there appear normally elliptic 2D tori linked to the transition stable–complex unstable of the family. These invariant tori were computed numerically in [38]. For other applications, see [39] and references therein.

The analytic approach to the problem relies on the computation of normal forms. The generic situation at the resonant periodic orbit is a non-semi-simple structure for the Jordan blocks of the monodromy matrix associated with the colliding characteristic multipliers. This normal form is a suspension of the normal form of a non-semi-simple equilibrium of a Hamiltonian system with two degrees of freedom having two equal frequencies. The normal form for this latter case first appeared in [55] (see also chapter 7 of [1]). For the detailed computation of the normal form around a $1:-1$ resonant periodic orbit we refer to [39, 43] (see also [7] for the extension to a $1:-1$ resonant invariant torus).

If we consider a one-parameter unfolding of the normal form, computed up to degree four, for the resonant equilibrium of the Hamiltonian system with two degrees of freedom described above, then the unfolded system undergoes the so-called *Hamiltonian Hopf bifurcation* (see [57]). This means that, when the parameter passes through the critical value, the corresponding family of equilibria suffers a complex destabilization and a one-parameter family of periodic orbits is created due to this collision of frequencies. This bifurcation can be *direct* (also called *supercritical*) or *inverse* (or *subcritical*). In the direct case, the bifurcated periodic orbits are elliptic, while in the inverse case parabolic and hyperbolic periodic orbits are also present through a centre-saddle bifurcation. The persistence of these bifurcated periodic orbits is not in question if a non-integrable perturbation is added to this construction.

For the critical periodic orbit of the three degrees of freedom Hamiltonian, the corresponding 'suspended' normal form up to degree four undergoes, generically, a *periodic Hamiltonian Hopf bifurcation*. Now it is not necessary to add an external parameter to the Hamiltonian, since the self energy of the system plays this rôle. In this context, the branching off of a biparametric family of two-dimensional invariant tori follows at once from the dynamics of this (integrable) normal form. In the direct case only normally elliptic tori unfold, while in the inverse case normally parabolic and hyperbolic tori are also present. The type of bifurcation is determined by the coefficients of the normal form.

However, this bifurcation pattern cannot be directly stated for the complete Hamiltonian, since the persistence of the invariant tori of the normal form when the remaining part of the system is taken into account is a problem involving small divisors. Hence, the question is whether some quasiperiodic solutions of the integrable part survive in the whole system, and we know there are chances for this to happen if this remainder is sufficiently small to be thought of as a perturbation.

This work tackles the persistence of the 2D-elliptic bifurcated tori in the direct periodic Hamiltonian Hopf bifurcation, following the approach introduced in [43]. More precisely, in theorem 3.1 we prove that there exists a two-parameter Cantor family of two-dimensional analytic elliptic tori branching off the resonant periodic orbit and give (asymptotic) quantitative estimates on the (Lebesgue) measure, in the parameter space, of the holes between invariant tori. Concretely, we show the typical 'condensation' phenomena of invariant tori of Kolmogorov–Arnold–Moser (KAM) theory: the measure of these holes goes to zero, as the values of the parameters approach those of the critical periodic orbit, faster than any algebraic order. However, for reasons we explain below, in this case we cannot obtain classical exponentially small estimates for this measure. Following a notation first introduced in [33], we call the union of this Cantor family of invariant tori an *invariant Cantor manifold*. Then, we also prove the Whitney-C^∞ smoothness of this 4D-Cantor manifold.

The existence of this invariant Cantor manifold of bifurcated tori also follows from [7, 10] (see also [23]). In [10], a very general methodology allowing to deal with real analytic nearly-integrable Hamiltonian systems (among other dynamical contexts) is developed, so that there is an invariant torus of the unperturbed system whose *normal linear part* (see section 4.3 for a precise definition) has multiple Floquet exponents, as it happens at the $1:-1$ resonance. Then, after introducing a suitable *universal unfolding* of the normal linear part at the resonant torus, the authors apply classical results of *parametrized KAM theory* to the extended system—the so-called *Broer–Huitema–Takens theory*, see [12]. With this setting, for a nearly full-measure Cantor set of parameters close to the critical ones, the persistence is shown, not only of the invariant tori of the unfolded integrable system but also of its corresponding linear normal part. The Whitney-C^∞ smoothness of this construction is also established.

This result is applied in [7] to the *direct quasiperiodic Hamiltonian Hopf bifurcation*—i.e. when the $1:-1$ resonance occurs at a n-dimensional torus of a Hamiltonian system with $(n+2)$ degrees of freedom—taking as a paradigmatic example the perturbed Lagrange top. More precisely, in [7] the existence of a Cantor family of $(n+1)$-dimensional elliptic invariant tori branching off the critical n-dimensional torus is shown. In addition, this paper also treats the complete quasiperiodic stratification around this resonant torus, by proving the existence of a Cantor family of n-dimensional tori containing the resonant one and of $(n+2)$-Lagrangian tori surrounding the $(n+1)$-dimensional bifurcated ones. Of course, for $n = 1$ we have the periodic Hopf bifurcation (but in this case the existence of a continuous family of 1D tori—periodic orbits—containing the critical one is straightforward).

It is worth comparing the results described above with theorem 3.1. First, we remark that we have selected the periodic Hamiltonian Hopf bifurcation instead of the quasiperiodic one because this latter complicates the periodic case in a way that it is not directly related to the singular bifurcation scenario we want to discuss in this paper. Hence, the context we have selected is the simplest one in which elliptic low-dimensional tori appear linked to a Hopf bifurcation in the Hamiltonian context. In order to extend the methods of this paper to the $(n+1)$-dimensional elliptic tori of the quasiperiodic Hopf bifurcation described above, a significant difference we mention is that, instead of a continuous family of periodic orbits undergoing complex destabilization, in the quasiperiodic case the corresponding family of n-dimensional tori holding the critical n-dimensional one only exists for a Cantor set of parameters. However, in this paper this continuous family of periodic orbits is not used to prove the existence of the 2D tori, but only to describe the transition between real and complex bifurcated tori (see theorem 3.1 and comments following theorem 4.1 for a more precise explanation).

Concerning the different methodologies followed in the proofs, the use in [10] of a universal unfolding of the linear normal part yields a general and elegant methodology for the study of nearly-integrable systems having a periodic orbit or a torus with multiple Floquet exponents. However, this approach needs the addition of extra parameters in order to guarantee the preservation of both the tori and the linear part. Some of these parameters can be introduced using natural variables of the Hamiltonian system, assuming that certain non-degeneracy conditions are fulfilled. In the case at hand, we need *one* external parameter to completely characterize the 2D-elliptic tori of the periodic Hopf bifurcation. Then, in order to ensure the existence of the family of bifurcated tori for a Hamiltonian free of parameters, one can apply the so-called *Herman's method*. Indeed, after an external parameter is introduced to have a complete unfolding of the system, then it can be eliminated, under very weak non-degeneracy conditions, by means of an appropriate technical result concerning *Diophantine approximations of dependent quantities* (see [47]). In this way, we can ensure almost full measure of the Cantor set of parameters for which the bifurcated 2D tori exist in the original parameter-less Hamiltonian system. We refer to [11, 21, 49, 51–53] for details on Herman's method.

However, it is not clear whether 'sharp' asymptotic estimates for the 'condensation' of tori can be obtained via Herman's approach, at least in a direct way. Herman's method is *optimal* in the sense that when we set the extra parameter to zero we completely characterize the invariant tori of the original system, but what are not optimal are the quantitative measure estimates applied to the normal form used in this paper (see theorem 4.1). In a few words, if one wanted to use Herman's method in order to obtain measure estimates as a function of the distance, R, to the critical periodic orbit, one would find that some of the quantities appearing in the technical results on Diophantine approximations of dependent quantities—estimates on the proximity of the frequency maps to those of the integrable system, the Diophantine constants and the size of the measure estimates one wishes to obtain for the resonant holes—that are usually assumed to be unrelated to the formulation of these results, are now R-dependent, in a way that strongly depends on the context we are dealing with.

In order to obtain such asymptotic estimates, in this paper we follow a different approach, which is an adaptation of the ideas introduced in [28] to the present close-to-resonant case.

To finish this section, let us summarize more precisely the most outstanding points of our approach and the main difficulties we have to face for proving theorem 3.1.

First of all, by means of normal forms we derive accurate approximations to the parametrizations of the bifurcated 2D-elliptic tori. We point out that the normal form associated with a Hamiltonian $1:-1$ resonance, computed at all orders, is generically divergent. Nevertheless, if we stop the normalizing process up to some finite order, the initial Hamiltonian is then cast (by means of a canonical transformation) into the sum of an integrable part plus a non-integrable remainder. For this integrable system, at any given order, the bifurcation pattern is the same as the one derived for the low-order normal form. We refer to [39] for a detailed analysis of the dynamics associated with this truncated normal form. Hence, a natural question is to ask for the 'optimal' order up to which this normal form must be computed to make—for a given distance, R, to the resonant periodic orbit—the remainder as small as possible (therefore both the order and the size of the remainder are given as a function of R). Thus, on the one hand, the smaller the asymptotic estimates on the remainder could be made, the worse the 'Diophantine constants' of the constructed invariant tori will be. On the other hand, the same estimates are translated into bounds for the relative measure of the complement of the Cantor set of parameters corresponding to invariant tori of the initial Hamiltonian system.

Moreover, when computing the normal form of a Hamiltonian around maximal dimensional tori, elliptic fixed points or normally elliptic periodic orbits or tori, there are (standard) results providing exponentially small estimates for the size of the remainder as a function of the distance, R, to the object (if the order of the normal form is chosen appropriately as a function of R). This fact leads to the classical exponentially small measure estimates in KAM theory (see, for instance, [8, 19, 27–29, 31]). However, for the periodic Hopf bifurcation, the non-semi-simple structure of the monodromy matrix at the critical orbit yields homological equations in the normal form computations that cannot be reduced to the diagonal form. This is an essential point, because when the homological equations are diagonal, only one 'small divisor' appears as a denominator of any coefficient of the solution. In contrast, in the non-semi-simple case, there are (at any order) some coefficients having as a denominator a small divisor raised up to the order of the corresponding monomial. This fact gives rise to very large 'amplification factors' in the normal form computations, which do not allow one to obtain exponentially small estimates for the remainder. In [40] it is proved that it decays with respect to R faster than any power of R, but with less sharp bounds than in the semi-simple case. This fact translates into poor asymptotic measure estimates for the bifurcated tori.

Once we have computed the invariant tori of the normal form, to prove the persistence of them in the complete system, we are faced with KAM methods for elliptic low-dimensional tori (see [11, 12, 20, 28, 29, 46]). More precisely, the proof resembles those on the existence of invariant tori when adding to a periodic orbit the excitations of its elliptic normal modes (compare [20,28,50]), but with the additional intricacies due to the present bifurcation scenario. The main difficulty in tackling this persistence problem has to do with the choice of suitable parameters to characterize the tori of the family along the iterative KAM process. In this case one has three frequencies to control, the two intrinsic (those of the quasiperiodic motion) and the normal one, but only two parameters (those of the family) to keep track of them. So, we are bound to deal with the so-called 'lack of parameters' problem for low-dimensional tori (see [11, 37, 51]). In this paper, instead of adding an external parameter to the system (in the aim of Herman's method), we select a suitable set of frequencies in order to label the bifurcated 2D tori and consider the remaining one as a function of the other two, even though some of the usual tricks for dealing with elliptic tori cannot be applied directly to the problem at hand, for the reasons explained below.

Indeed, when applying KAM techniques for invariant tori of Hamiltonian systems, one way to proceed is to set a diffeomorphism between the intrinsic frequencies and the 'parameter space' of the family of tori (typically the actions). In this way, in the case of elliptic low-dimensional tori, the normal frequencies can be expressed as a function of the intrinsic ones. Under these assumptions, the standard non-degeneracy conditions on the normal frequencies require that the denominators of the KAM process, which depend on the normal and on the intrinsic frequencies, 'move' as a function of the latter. Assuming these *transversality* conditions, the Diophantine ones can be fulfilled at each step of the KAM iterative process. Unfortunately, in the current context these transversality conditions are not defined at the critical orbit, due to the strong degeneracy of the problem. In a few words, the elliptic invariant tori we study are too close to parabolic. This catch is worked out taking as the vector of *basic frequencies* (those labelling the tori) not the intrinsic ones, say $\Omega = (\Omega_1, \Omega_2)$, but the vector $\Lambda = (\mu, \Omega_2)$, where μ is the normal frequency of the torus. Then we put the other (intrinsic) frequency as a function of Λ, i.e. $\Omega_1 = \Omega_1(\Lambda)$. With this parametrization, the denominators of the KAM process move with Λ even if we are close to the resonant periodic orbit. An alternative possibility is to fix the frequency ratio $[\mu : \Omega_1 : \Omega_2]$ (see further comments in remark 5.1).

Another difficulty we have to face refers to the computation of the sequence of canonical transformations of the KAM scheme. At any step of this iterative process we compute the corresponding canonical transformation by means of the Lie method. Typically in the KAM context, the (homological) equations verified by the generating function of this transformation are coupled through a triangular structure, so we can solve them recursively. However, due to the aforementioned proximity to parabolic, in the present case some equations—corresponding to the average of the system with respect to the angles of the tori—become simultaneously coupled and have to be solved all together. Then the resolution of the homological equations becomes a little more tricky, specially with regard to the verification of the non-degeneracy conditions required to solve them. This is the main price we paid for not using the approach of [10] based on universal unfoldings of matrices.

This work is organized as follows. We begin fixing the notation and introducing several definitions in section 2. In section 3 we formulate theorem 3.1, which constitutes the main result of the paper. Section 4 is devoted to reviewing some previous results about the normal form around a $1 : -1$ resonant periodic orbit (both from the qualitative and quantitative point of view). The proof of theorem 3.1 is given in section 5, whilst in appendix A we compile some technical results used throughout the text.

2. Basic notation and definitions

Given a complex vector $u \in \mathbb{C}^n$, we denote by $|u|$ its supremum norm, $|u| = \sup_{1\leqslant i\leqslant n}\{|u_i|\}$. We extend this notation to any matrix $A \in \mathbb{M}_{r,s}(\mathbb{C})$, so that $|A|$ means the induced matrix norm. Similarly, we write $|u|_1 = \sum_{i=1}^n |u_i|$ for the absolute norm of a vector and $|u|_2$ for its Euclidean norm. We denote by u^* and A^* the transpose vector and matrix, respectively. As usual, for any $u, v \in \mathbb{C}^n$, their bracket $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$ is the inner product of $\mathbb{C}^n$. Moreover, $\lfloor\cdot\rfloor$ stands for the integer part of a real number.

We deal with analytic functions $f = f(\theta, x, I, y)$ defined in the domain[1]

$$\mathcal{D}_{r,s}(\rho, R) = \{(\theta, x, I, y) \in \mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^r \times \mathbb{C}^s : |\mathrm{Im}\,\theta| \leqslant \rho,\ |(x, y)| \leqslant R,\ |I| \leqslant R^2\}, \tag{1}$$

for some integers r, s and some $\rho > 0$, $R > 0$. These functions are 2π-periodic in θ and take values on $\mathbb{C}$, $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$. By expanding f in the Taylor–Fourier series (we use multi-index notation throughout the paper),

$$f = \sum_{(k,l,m)\in\mathbb{Z}^r\times\mathbb{Z}_+^r\times\mathbb{Z}_+^{2s}} f_{k,l,m} \exp(\mathrm{i}\langle k, \theta\rangle) I^l z^m, \tag{2}$$

where $z = (x, y)$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we introduce the weighted norm

$$|f|_{\rho,R} = \sum_{k,l,m} |f_{k,l,m}| \exp(|k|_1\rho) R^{2|l|_1+|m|_1}. \tag{3}$$

We observe that $|f|_{\rho,R} < +\infty$ implies that f is analytic in the interior of $\mathcal{D}_{r,s}(\rho, R)$ and bounded up to the boundary. Conversely, if f is analytic in a neighbourhood of $\mathcal{D}_{r,s}(\rho, R)$, then $|f|_{\rho,R} < +\infty$. Moreover, we point out that $|f|_{\rho,R}$ is an upper bound for the supremum norm of f in $\mathcal{D}_{r,s}(\rho, R)$. Some of the properties of this norm have been surveyed in appendix A.1. These properties are very similar to the corresponding ones for the supremum norm. We work

[1] We point out that, depending on the context, the set $\mathcal{D}_{r,s}(\rho, R)$ is used with $r = 1$, $s = 2$ or with $r = 2$, $s = 1$.

with weighted norms instead of the supremum norm because some estimates become simpler with them, especially those on small divisors. Several examples of the use of these norms can be found in [19, 28, 40]. Alternatively, one can work with the supremum norm and use the estimates of Rüssmann on small divisors (see [48]).

For a complex-valued function $f = f(\theta, x, I, y)$ we use Taylor expansions of the form

$$f = \mathrm{a}(\theta) + \langle \mathrm{b}(\theta), z\rangle + \langle \mathrm{c}(\theta), I\rangle + \tfrac{1}{2}\langle z, \mathrm{B}(\theta)z\rangle + \langle I, \mathrm{E}(\theta)z\rangle + \tfrac{1}{2}\langle I, \mathrm{C}(\theta)I\rangle + \mathrm{F}(\theta, x, I, y), \tag{4}$$

with $\mathrm{B}^* = \mathrm{B}$, $\mathrm{C}^* = \mathrm{C}$ and F holding the higher order terms with respect to (z, I). From (4) we introduce the notation $[f]_0 = \mathrm{a}$, $[f]_z = \mathrm{b}$, $[f]_I = \mathrm{c}$, $[f]_{z,z} = \mathrm{B}$, $[f]_{I,z} = \mathrm{E}$, $[f]_{I,I} = \mathrm{C}$ and $[f] = \mathrm{F}$.

The coordinates $(\theta, x, I, y) \in \mathcal{D}_{r,s}(\rho, R)$ are canonical through the symplectic form $\mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}x \wedge \mathrm{d}y$. Hence, given scalar functions $f = f(\theta, x, I, y)$ and $g = g(\theta, x, I, y)$, we define their *Poisson bracket* by

$$\{f, g\} = (\nabla f)^* \mathcal{J}_{r+s} \nabla g,$$

where ∇ is the gradient with respect to (θ, x, I, y) and $\mathcal{J}_n$ the standard symplectic $2n \times 2n$ matrix. If $\Psi = \Psi(\theta, x, I, y)$ is a canonical transformation, close to the identity, then we consider the following expression of Ψ (according to its natural vector components),

$$\Psi = \mathrm{Id} + (\Theta, \mathcal{X}, \mathcal{I}, \mathcal{Y}), \qquad \mathcal{Z} = (\mathcal{X}, \mathcal{Y}). \tag{5}$$

To generate such canonical transformations we mainly use the *Lie series method*. Thus, given a Hamiltonian $H = H(\theta, x, I, y)$ we denote by Ψ_t^H the flow time t of the corresponding vector field, $\mathcal{J}_{r+s}\nabla H$. We observe that if $\mathcal{J}_{r+s}\nabla H$ is 2π-periodic in θ, then also is $\Psi_t^H - \mathrm{Id}$.

Let $f = f(\theta)$ be a 2π-periodic function defined in the r-dimensional complex strip

$$\Delta_r(\rho) = \{\theta \in \mathbb{C}^r : |\mathrm{Im}\,\theta| \leqslant \rho\}. \tag{6}$$

If we expand f in the Fourier series, $f = \sum_{k\in\mathbb{Z}^r} f_k \exp(\mathrm{i}\langle k, \theta\rangle)$, we observe that $|f|_{\rho,0}$ gives the weighted norm of f in $\Delta_r(\rho)$. Moreover, given $N \in \mathbb{N}$, we consider the following truncated Fourier expansions,

$$f_{<N,\theta} = \sum_{|k|_1<N} f_k \exp(\mathrm{i}\langle k, \theta\rangle), \qquad f_{\geqslant N,\theta} = f - f_{<N,\theta}. \tag{7}$$

Notation (7) can also be extended to $f = f(\theta, x, I, y)$. Furthermore, we also introduce

$$L_\Omega f = \sum_{j=1}^{r} \Omega_j \partial_{\theta_j} f, \qquad \langle f\rangle_\theta = \frac{1}{(2\pi)^r}\int_{\mathbb{T}^r} f(\theta)\,\mathrm{d}\theta, \qquad \{f\}_\theta = f - \langle f\rangle_\theta, \tag{8}$$

where $\Omega \in \mathbb{R}^r$ and $\mathbb{T}^r = (\mathbb{R}/2\pi\mathbb{Z})^r$. We refer to $\langle f\rangle_\theta$ as the *average* of f.

Given an analytic function $f = f(u)$ defined for $u \in \mathbb{C}^n$, $|u| \leqslant R$, we consider its Taylor expansion around the origin, $f(u) = \sum_{m\in\mathbb{Z}_+^n} f_m u^m$, and define $|f|_R = \sum_m |f_m| R^{|m|_1}$.

Let $f = f(\phi)$ be a function defined for $\phi \in \mathcal{A} \subset \mathbb{C}^n$. For this function we denote its supremum norm and its Lipschitz constant by

$$|f|_{\mathcal{A}} = \sup_{\phi\in\mathcal{A}} |f(\phi)|, \qquad \mathrm{Lip}_{\mathcal{A}}(f) = \sup\left\{\frac{|f(\phi') - f(\phi)|}{|\phi' - \phi|} : \phi, \phi' \in \mathcal{A},\ \phi \neq \phi'\right\}$$

Moreover, if $f = f(\theta, x, I, y; \phi)$ is a family of functions defined in $\mathcal{D}_{r,s}(\rho, R)$, for any $\phi \in \mathcal{A}$, we denote by $|f|_{\mathcal{A},\rho,R} = \sup_{\phi\in\mathcal{A}} |f(\cdot;\phi)|_{\rho,R}$.

Finally, given $\sigma > 0$, one defines the *complex σ-widening* of the set $\mathcal{A}$ as

$$\mathcal{A} + \sigma = \bigcup_{z\in\mathcal{A}} \{z' \in \mathbb{C}^n : |z - z'| \leqslant \sigma\}, \tag{9}$$

i.e. $\mathcal{A} + \sigma$ is the union of all (complex) balls of radius σ (in the norm $|\cdot|$) centred at points of $\mathcal{A}$.

3. Formulation of the main result

Let us consider a three degrees of freedom real analytic Hamiltonian system $\mathcal{H}$ with a $1:-1$ resonant periodic orbit. We assume that we have a system of symplectic coordinates specially suited for this orbit, so that the phase space is described by $(\theta, x, I, y) \in \mathbb{T}^1 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, being $x = (x_1, x_2)$ and $y = (y_1, y_2)$, endowed with the 2-form $\mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}x \wedge \mathrm{d}y$. In this reference system we want the periodic orbit to be given by the circle $I = 0$, $x = y = 0$. Such (local) coordinates can always be found for a given periodic orbit (see [13, 14, 30] for an explicit example). In addition, a (symplectic) Floquet transformation is performed to reduce to constant coefficients the quadratic part of the Hamiltonian with respect to the *normal directions* (x, y) (see [43]). If the resonant eigenvalues of the monodromy matrix of the critical orbit are non-semi-simple, the Hamiltonian expressed in the new variables can be written as[2]

$$\mathcal{H}(\theta, x, I, y) = \omega_1 I + \omega_2(y_1 x_2 - y_2 x_1) + \tfrac{1}{2}(y_1^2 + y_2^2) + \hat{\mathcal{H}}(\theta, x, I, y), \qquad (10)$$

where ω_1 is the angular frequency of the periodic orbit and ω_2 its (only) normal frequency, so that its non-trivial characteristic multipliers are $\{\lambda, \lambda, 1/\lambda, 1/\lambda\}$, with $\lambda = \exp(2\pi \mathrm{i}\omega_2/\omega_1)$. The function $\hat{\mathcal{H}}$ is 2π-periodic in θ, holds the higher order terms in (x, I, y) and can be analytically extended to a complex neighbourhood of the periodic orbit. From now on, we set $\mathcal{H}$ to be our initial Hamiltonian.

To describe the dynamics of $\mathcal{H}$ around the critical orbit we use normal forms. A detailed computation of the normal form for a $1:-1$ resonant periodic orbit can be found in [7, 39, 43]. The only (generic) non-resonant condition required to carry out this normalization (at any order) is that $\omega_1/\omega_2 \notin \mathbb{Q}$, which is usually referred to as *irrational collision*.

The normalized Hamiltonian of (10) up to 'degree four' in (x, I, y) looks like

$$\mathcal{Z}_2(x, I, y) = \omega_1 I + \omega_2 L + \tfrac{1}{2}(y_1^2 + y_2^2) + \tfrac{1}{2}(aq^2 + bI^2 + cL^2) + dqI + eqL + fIL, \qquad (11)$$

where $q = (x_1^2 + x_2^2)/2$, $L = y_1 x_2 - y_2 x_1$ and $a, b, c, d, e, f \in \mathbb{R}$. As usual, the contribution of the action I to the degree is counted twice. Now, writing the Hamilton equations of $\mathcal{Z}_2$, it is easy to realize that the manifold $x = y = 0$ is foliated by a family of periodic orbits, parametrized by I, that contains the critical one. By assuming irrational collision, it is clear that—applying the Lyapunov centre theorem, see [54]—this family also exists (locally) for the full system (10). The (non-degeneracy) condition that determines the transition from stability to complex instability of this family is $d \neq 0$. Moreover, the direct or inverse character of the bifurcation is defined in terms of the sign of a and, for our concerns, $a > 0$ implies direct bifurcation. Hence, in the forthcoming we shall assume that $d \neq 0$ and $a > 0$.

We refer to [7, 23] for a detailed description of the singularity theory aspects of the direct periodic Hamiltonian Hopf bifurcation, as an extension of the results for the classical Hamiltonian Hopf bifurcation (see [18, 57]). In a few words, as the normal form (11) is integrable, we can consider the so-called *energy–momentum mapping* EM, given by $\mathrm{EM} = (I, L, \mathcal{Z}_2)$. This map gives rise to a stratification of the six-dimensional phase space into invariant tori. Indeed, for any regular value of this mapping its pre-image defines a three-dimensional (Lagrangian) invariant torus, and a singular value of EM corresponds to a 'pinched' three-dimensional torus (i.e. a hyperbolic periodic orbit and its stable and unstable invariant manifolds), an elliptic 2D torus or an elliptic periodic orbit.

Once a direct periodic Hopf bifurcation is set, we can establish for the dynamics of $\mathcal{Z}_2$ and, in fact, for the dynamics of the truncated normal form up to an arbitrary order, the existence of a two-parameter family of two-dimensional elliptic tori branching off the resonant periodic orbit. A detailed analysis of the (integrable) dynamics associated with this normal form, up

[2] Nevertheless, to achieve this form, an involution in time may yet be necessary. See [39].

to an arbitrary order, can be found in [39, 43]. Of course, due to the small divisors of the problem, it is not possible to expect full persistence of this family in the complete Hamiltonian system (10), but only a Cantor family of two-dimensional tori. For a proof of the existence of this invariant Cantor manifold see [43] and for a complete discussion of the 'Cantorization' of the 'quasiperiodic stratification' described above we refer to [7].

The precise result we have obtained about the persistence of this family is stated as follows and constitutes the main result of the paper.

Theorem 3.1. *We assume that the real analytic Hamiltonian $\mathcal{H}$ in (10) is defined in the complex domain $\mathcal{D}_{1,2}(\rho_0, R_0)$, for some $\rho_0 > 0$, $R_0 > 0$, and that the weighted norm $|\mathcal{H}|_{\rho_0, R_0}$ is finite. Moreover, we also assume that the (real) coefficients a and d of its low-order normal form $\mathcal{Z}_2$ in (11) verify $a > 0$, $d \neq 0$, and that the vector $\omega = (\omega_1, \omega_2)$ satisfies the Diophantine condition*[3]

$$|\langle k, \omega\rangle| \geqslant \gamma |k|_1^{-\tau}, \qquad \forall\, k \in \mathbb{Z}^2 \setminus \{0\}, \tag{12}$$

for some $\gamma > 0$ and $\tau > 1$. Then, we have the following.

(i) The $1:-1$ resonant periodic orbit $I = 0$, $x = y = 0$ of $\mathcal{H}$ is embedded into a one-parameter family of periodic orbits having a transition from stability to complex instability at this critical orbit.

(ii) There exists a Cantor set $\mathcal{E}^{(\infty)} \subset \mathbb{R}^+ \times \mathbb{R}$ and a function $\Omega_1^{(\infty)} : \mathcal{E}^{(\infty)} \to \mathbb{R}$ such that, for any $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$, the Hamiltonian system $\mathcal{H}$ has an analytic two-dimensional elliptic invariant torus—with a vector of intrinsic frequencies $\Omega(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$ and normal frequency μ—branching off the critical periodic orbit. However, for some values of Λ this torus is complex (i.e. a torus lying on the complex phase space but carrying out quasiperiodic motion for real time).

(iii) The 'density' of the set $\mathcal{E}^{(\infty)}$ becomes almost one as we approach the resonant periodic orbit. Indeed, there exist constants $c^* > 0$ and $\bar{c}^* > 0$ such that, if we define

$$\mathcal{V}(R) := \{\Lambda = (\mu, \Omega_2) \in \mathbb{R}^2 : 0 < \mu \leqslant c^* R,\ |\Omega_2 - \omega_2| \leqslant c^* R\}$$

and $\mathcal{E}^{(\infty)}(R) = \mathcal{E}^{(\infty)} \cap \mathcal{V}(R)$, then, for any given $0 < \alpha < 1/19$, there is $\breve{R}^* = \breve{R}^*(\alpha)$ such that

$$\operatorname{meas}\left(\mathcal{V}(R) \setminus \mathcal{E}^{(\infty)}(R)\right) \leqslant \bar{c}^* (M^{(0)}(R))^{\alpha/4}, \tag{13}$$

for any $0 < R \leqslant \breve{R}^*$. Here, meas *stands for the Lebesgue measure of $\mathbb{R}^2$ and the expression $M^{(0)}(R)$, which is defined precisely in the statement of theorem 4.1, goes to zero faster than any power of R (although it is not exponentially small in R).*

(iv) There exists a real analytic function $\tilde{\Omega}_2$, with $\tilde{\Omega}_2(0) = \omega_2$, such that the curves $\gamma_1(\eta) = (2\eta, \eta + \tilde{\Omega}_2(\eta^2))$ and $\gamma_2(\eta) = (2\eta, -\eta + \tilde{\Omega}_2(\eta^2))$ locally separate between the parameters $\Lambda \in \mathcal{E}^{(\infty)}$ giving rise to real or complex tori. Indeed, if $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$ and $\mu = 2\eta > 0$, then real tori are those with $-\eta + \tilde{\Omega}_2(\eta^2) < \Omega_2 < \eta + \tilde{\Omega}_2(\eta^2)$. The meaning of the curves γ_1 and γ_2 are that their graphs represent, in the Λ space, the periodic orbits of the family (i), but only those on the stable side of the transition. For a given $\eta > 0$, the periodic orbit labelled by $\gamma_1(\eta)$ is identified by the one labelled by $\gamma_2(\eta)$, $\eta + \tilde{\Omega}_2(\eta^2)$ and $-\eta + \tilde{\Omega}_2(\eta^2)$ being the two normal frequencies of the orbit ($\eta = 0$ corresponds to the critical one).

[3] The Lebesgue measure of the set of values $\omega \in \mathbb{R}^2$ for which condition (12) is not fulfilled is zero (see [34], appendix 4).

(v) The function $\Omega_1^{(\infty)} : \mathcal{E}^{(\infty)} \to \mathbb{R}$ *is* C^∞ *in the sense of Whitney. Moreover, for each* $\Lambda \in \mathcal{E}^{(\infty)}$*, the following Diophantine conditions are fulfilled by the intrinsic frequencies and the normal one of the corresponding torus:*

$$|\langle k, \Omega^{(\infty)}(\Lambda)\rangle + \ell\mu| \geqslant (M^{(0)}(R))^{\alpha/2}|k|_1^{-\tau}, \qquad k \in \mathbb{Z}^2, \quad \ell \in \{0, 1, 2\}, \quad |k|_1 + \ell \neq 0.$$

(vi) Let $\breve{\mathcal{E}}^{(\infty)}$ *be the subset of* $\mathcal{E}^{(\infty)}$ *corresponding to real tori. There is a function* $\Phi^{(\infty)}(\theta, \Lambda)$*, defined as* $\Phi^{(\infty)} : \mathbb{T}^2 \times \breve{\mathcal{E}}^{(\infty)} \to \mathbb{T} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$*, analytic in* θ *and Whitney-*C^∞ *with respect to* Λ*, giving a parametrization of the Cantorian four-dimensional manifold defined by the real two-dimensional invariant tori of* $\mathcal{H}$*, branching off the critical periodic orbit. Precisely, for any* $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$*, the function* $\Phi^{(\infty)}(\cdot, \Lambda)$ *gives a parametrization of the corresponding two-dimensional invariant torus of* $\mathcal{H}$*, in such a way that the pull-back of the dynamics on the torus to the variable* θ *is a linear quasiperiodic flow. Thus, for any* $\theta^{(0)} \in \mathbb{T}^2$*,* $t \in \mathbb{R} \mapsto \Phi^{(\infty)}(\Omega(\Lambda) \cdot t + \theta^{(0)}, \Lambda)$ *is a solution of the Hamilton equations of* $\mathcal{H}$*. Moreover,* $\Phi^{(\infty)}$ *can be extended to a smooth function of* $\mathbb{T}^2 \times \mathbb{R}^2$*—analytic in* θ *and* C^∞ *with respect to* Λ*.*

Remark 3.1. In this result we prove Whitney-C^∞ smoothness of the functions $\Phi^{(\infty)}$ and $\Omega_1^{(\infty)}$ with respect to Λ, which are obtained as a limit of sequences of analytic approximations (see section 5.14). Furthermore, using the super-exponential estimates on the speed of convergence of these sequences (see (117)) and applying the adaptation of the *inverse approximation lemma* proved in [58], Whitney–Gevrey smoothness of these limit functions might be achieved.

Remark 3.2. There is almost no difference in studying the persistence of elliptic tori in the inverse case using the approach of the paper for the direct case. For hyperbolic tori, the same methodology of the paper also works, only taking into account that now instead of the normal frequency we have to use as a parameter the real normal eigenvalue. Thus, in the hyperbolic case we can also use the iterative KAM scheme described in section 5.3, with the only difference that some of the divisors, appearing when solving the homological equations (eq1) and (eq2), are not 'small divisors' at all, because their real part has a uniform lower bound in terms of this normal eigenvalue. This fact simplifies a lot the measure estimates of the surviving tori. However, the final asymptotic measure estimates in the hyperbolic case will be of the same form as in (13), perhaps with a better (greater) exponent α. The parabolic case requires a different approach and it is not covered by this paper. We refer to example 4.5 of [23] for the proof of the persistence of these parabolic invariant tori (using the results of [9, 22]) and for a complete treatment of the inverse case.

Remark 3.3. It seems very feasible to obtain analogous asymptotic measure estimates for the three-dimensional (Lagrangian) tori surrounding the 2D-bifurcated ones. Nevertheless, to do that it is necessary to derive first the parametrizations of the unperturbed three-dimensional tori of the normal form up to an arbitrary order and of the corresponding three-dimensional vector of intrinsic frequencies (see remark 4.3).

Remark 3.4. In a very general setting, we can consider a direct quasiperiodic Hamiltonian Hopf bifurcation in $n + p + q + 2$ degrees of freedom, with a $1 : -1$ resonant torus of dimension n having a normal linear part with p (non-resonant) elliptic and q hyperbolic directions. In this case, we obtain a $(n + 1)$-parameter family of $(n + 1)$-dimensional bifurcated tori linked to this Hopf scenario, having a linear part with $p + 1$ elliptic and q hyperbolic directions. Asymptotic measure estimates such as those given in (13) can also be gleaned for such tori, by combining the techniques of this paper with the standard methods for dealing with asymptotic measure estimates close to non-resonant invariant tori as developed in [28]. However, to do that one must first generalize the quantitative estimates on the $1 : -1$ resonant normal form performed in [40] to the case of a $1:-1$ resonant n-dimensional torus. In addition, by taking also into account the ideas of [28] for the treatment of elliptic normal modes, one can prove similar asymptotic results for the existence of Cantor families of lower dimensional invariant tori of higher dimension.

The proof of theorem 3.1 extends to the end of this paper.

4. Previous results

In this section we review some previous results we use to carry out the proof of theorem 3.1. Concretely, in section 4.1 we discuss precisely what the normal form around a $1:-1$ resonant periodic orbit looks like and give, as a function of the distance to the critical orbit, quantitative estimates on the remainder of this normal form. In sections 4.2 and 4.3 we identify the family of 2D-bifurcated tori of the normal form, branching off the critical orbit, and its (linear) normal behaviour.

4.1. Quantitative normal form

Our first step is to compute the normal form of $\mathcal{H}$ in (10) up to a suitable order. This order is chosen to minimize (as much as possible) the size of the non-integrable remainder of the normal form. Hence, for any $R > 0$ (small enough), we consider a neighbourhood of 'size' R around the critical periodic orbit (see (1)) and select the normalizing order, $r_{\rm opt}(R)$, so that the remainder of the normal form of $\mathcal{H}$ up to degree $r_{\rm opt}(R)$ becomes as small as possible in this neighbourhood. As we have pointed out before, for an elliptic non-resonant periodic orbit (for a Diophantine vector of frequencies) it is possible to select this order so that the remainder becomes exponentially small in R. In the present resonant setting, the non-semi-simple character of the homological equations leads to poor estimates for the remainder. The following result, that can be derived from [40], states the normal form up to 'optimal' order and the bounds for the corresponding remainder.

Theorem 4.1. *With the same hypotheses as theorem 3.1. Given any $\varepsilon > 0$ and $\sigma > 1$, both fixed, there exists $0 < R^* < 1$ such that, for any $0 < R \leqslant R^*$, there is a real analytic canonical diffeomorphism $\hat{\Psi}^{(R)}$ verifying the following.*

(i) $\hat{\Psi}^{(R)} : \mathcal{D}_{1,2}(\sigma^{-2}\rho_0/2, R) \to \mathcal{D}_{1,2}(\rho_0/2, \sigma R)$.

(ii) If $\hat{\Psi}^{(R)} - \mathrm{Id} = (\hat{\Theta}^{(R)}, \hat{\mathcal{X}}^{(R)}, \hat{\mathcal{I}}^{(R)}, \hat{\mathcal{Y}}^{(R)})$, then all the components are 2π-periodic in θ and satisfy

$$|\hat{\Theta}^{(R)}|_{\sigma^{-2}\rho_0/2,R} \leqslant (1-\sigma^{-2})\rho_0/2, \qquad |\hat{\mathcal{I}}^{(R)}|_{\sigma^{-2}\rho_0/2,R} \leqslant (\sigma^2-1)R^2,$$
$$|\hat{\mathcal{X}}_j^{(R)}|_{\sigma^{-2}\rho_0/2,R} \leqslant (\sigma-1)R, \qquad |\hat{\mathcal{Y}}_j^{(R)}|_{\sigma^{-2}\rho_0/2,R} \leqslant (\sigma-1)R, \qquad j=1,2.$$

(iii) The transformed Hamiltonian by the action of $\hat{\Psi}^{(R)}$ takes the form

$$\mathcal{H}\circ\hat{\Psi}^{(R)}(\theta,x,I,y) = \mathcal{Z}^{(R)}(x,I,y) + \mathcal{R}^{(R)}(\theta,x,I,y), \tag{14}$$

where $\mathcal{Z}^{(R)}$ (the normal form) is an integrable Hamiltonian system which looks like

$$\mathcal{Z}^{(R)}(x,I,y) = \mathcal{Z}_2(x,I,y) + \tilde{Z}^{(R)}(x,I,y), \tag{15}$$

where $\mathcal{Z}_2$ is given by (11) and $\tilde{Z}^{(R)}(x,I,y) = Z^{(R)}(q,I,L/2)$, with $q = (x_1^2+x_2^2)/2$ and $L = y_1x_2 - x_1y_2$. The function $Z^{(R)}(u_1,u_2,u_3)$ is analytic around the origin, with the Taylor expansion starting at degree three. More precisely, $Z^{(R)}(u_1,u_2,u_3)$ is a polynomial of degree less than or equal to $\lfloor r_{\rm opt}(R)/2\rfloor$, except by the affine part on u_1 and u_3, which allows a general Taylor series expansion on u_2. The remainder $\mathcal{R}^{(R)}$ contains terms in (x,I,y) of higher order than 'the polynomial part' of $\mathcal{Z}^{(R)}$, all of them being of $\mathrm{O}_3(x,y)$.

(iv) The expression $r_{\mathrm{opt}}(R)$ *is given by*

$$r_{\mathrm{opt}}(R) := 2 + \left\lfloor \exp\left(W\left(\log\left(\frac{1}{R^{1/(\tau+1+\varepsilon)}}\right)\right)\right)\right\rfloor, \tag{16}$$

with $W : (0, +\infty) \to (0, +\infty)$ *defined from the equation* $W(z)\exp(W(z)) = z$.

(v) $\mathcal{R}^{(R)}$ *satisfies the bound*

$$|\mathcal{R}^{(R)}|_{\sigma^{-2}\rho_0/2,R} \leqslant M^{(0)}(R) := R^{r_{\mathrm{opt}}(R)/2}. \tag{17}$$

In particular, $M^{(0)}(R)$ *goes to zero with R faster than any algebraic order, that is*

$$\lim_{R\to 0^+} \frac{M^{(0)}(R)}{R^n} = 0, \qquad \forall n \geqslant 1.$$

(vi) There exists a constant $\tilde{c}$ *independent of R (but depending on* ε *and* σ*) such that*

$$|\mathcal{Z}^{(R)}|_{0,R} \leqslant |\mathcal{H}|_{\rho_0,R_0}, \qquad |\tilde{\mathcal{Z}}^{(R)}|_{0,R} \leqslant \tilde{c}R^6. \tag{18}$$

Remark 4.1. The function W corresponds to the principal branch of a special function $W : \mathbb{C} \to \mathbb{C}$ known as the Lambert W function. A detailed description of its properties can be found in [17].

Actually, the full statement of theorem 4.1 is not explicitly contained in [40], but can be easily gleaned from the paper. Let us describe which are the new features we are talking about.

First, we have modified the action of the transformation $\hat{\Psi}^{(R)}$ so that the family of periodic orbits of $\mathcal{H}$, in which the critical orbit is embedded, and its normal (Floquet) behaviour are fully described (locally) by the normal form $\mathcal{Z}^{(R)}$ of (15). Thus, the fact that the remainder $\mathcal{R}^{(R)}$ is of $\mathrm{O}_3(x, y)$ implies that neither the family of periodic orbits nor its Floquet multipliers change in (14) from those of $\mathcal{Z}^{(R)}$ (see sections 4.2 and 4.3). To achieve this, we are forced to work not only with a polynomial expression for the normal form $\mathcal{Z}^{(R)}$ (as done in [40]), but to allow a general Taylor series expansion on I for the coefficients of the affine part of the expansion of $\mathcal{Z}^{(R)}$ in powers of q and L. For this purpose, we have to extend the normal form criteria used in [40]. We do not plan to give here full details on these modifications, but we are going to summarize the main ideas below.

Let us consider the initial Hamiltonian $\mathcal{H}$ in (10). Then we start by applying a *partial normal form* process to it in order to reduce the remainder to $\mathrm{O}_3(x, y)$ and to arrange the affine part of the normal form in q and L. After this process, the family of periodic orbits of $\mathcal{H}$ and its Floquet behaviour remain the same if we compute them either in the complete transformed system or in the truncated one when removing the $\mathrm{O}_3(x, y)$ remainder. We point out that the divisors appearing in this (partial) normal form are $k\omega_1 + l\omega_2$, with $k \in \mathbb{Z}$ and $l \in \{0, \pm 1, \pm 2\}$ (excluding the case $k = l = 0$). As we are assuming irrational collision, these divisors are not 'small divisors' at all, because all of them are uniformly bounded from below and go to infinity with k. Hence, we can ensure convergence of this normalizing process in a neighbourhood of the periodic orbit.

After we carry out this convergent (partial) normal form scheme on $\mathcal{H}$, we apply the result of [40] to the resulting system. In this way we establish the quantitative estimates, as a function of R, for the normal form up to 'optimal order'. It is easy to realize that the normal form procedure of [40] does not 'destroy' the $\mathrm{O}_3(x, y)$ structure of the remainder $\mathcal{R}^{(R)}$.

However, we want to emphasize that the particular structure for the normal form $\mathcal{Z}^{(R)}$ stated in theorem 4.1 is not necessary to apply KAM methods. We can prove the existence of the (Cantor) bifurcated family of 2D tori by using the polynomial normal form of [40]. The reason motivating the modifying of the former normal form is only to characterize easily which bifurcated tori are real tori as stated in point (iv) of the statement of theorem 3.1 (for details, see remark 4.2 and section 5.13).

The second remark in theorem 4.1 refers to the bound on $\tilde{Z}^{(R)}$ given in the last point of the statement, that neither is explicitly contained in [40]. Again, it can be easily gleaned from the paper. However, there is also the chance to derive it by hand from the bound on $\mathcal{Z}^{(R)}$ and its particular structure. This is done in appendix A.2.

4.2. Bifurcated family of 2D tori of the normal form

It turns out that the normal form $\mathcal{Z}^{(R)}$ is integrable, but in this paper we are only concerned with the two-parameter family of bifurcated 2D-invariant tori associated with this Hopf scenario. See [39] for a full description of the dynamics. To easily identify this family, we introduce new (canonical) coordinates $(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}$, with the 2-form $\mathrm{d}\phi \wedge \mathrm{d}J + \mathrm{d}q \wedge \mathrm{d}p$, defined through the change

$$\begin{aligned} \theta &= \phi_1, & x_1 &= \sqrt{2q}\cos\phi_2, & y_1 &= -\frac{J_2}{\sqrt{2q}}\sin\phi_2 + p\sqrt{2q}\cos\phi_2, \\ I &= J_1, & x_2 &= -\sqrt{2q}\sin\phi_2, & y_2 &= -\frac{J_2}{\sqrt{2q}}\cos\phi_2 - p\sqrt{2q}\sin\phi_2, \end{aligned} \tag{19}$$

which casts the Hamiltonian (14) into (dropping the superindex (R))

$$\breve{\mathcal{H}}(\phi, q, J, p) = \breve{\mathcal{Z}}(q, J, p) + \breve{\mathcal{R}}(\phi, q, J, p), \tag{20}$$

where

$$\begin{aligned} \breve{\mathcal{Z}}(q, J, p) = {} & \langle \omega, J \rangle + qp^2 + \frac{J_2^2}{4q} + \frac{1}{2}(aq^2 + bJ_1^2 + cJ_2^2) \\ & + dqJ_1 + eqJ_2 + fJ_1J_2 + Z(q, J_1, J_2/2). \end{aligned} \tag{21}$$

Let us consider the Hamilton equations of $\breve{\mathcal{Z}}$:

$$\begin{aligned} \dot{\phi}_1 &= \omega_1 + bJ_1 + dq + fJ_2 + \partial_2 Z(q, J_1, J_2/2), & \dot{J}_1 &= 0, \\ \dot{\phi}_2 &= \omega_2 + \frac{J_2}{2q} + cJ_2 + eq + fJ_1 + \frac{1}{2}\partial_3 Z(q, J_1, J_2/2), & \dot{J}_2 &= 0, \\ \dot{p} &= -p^2 + \frac{J_2^2}{4q^2} - aq - dJ_1 - eJ_2 - \partial_1 Z(q, J_1, J_2/2), & \dot{q} &= 2qp. \end{aligned}$$

The next result sets precisely the bifurcated family of 2D tori of $\breve{\mathcal{Z}}$ (and hence of $\mathcal{Z}$).

Theorem 4.2. *With the same notation as theorem 4.1. If $d \neq 0$, there exists a real analytic function $\mathbb{I}(\xi, \eta)$ defined in $\Gamma \subset \mathbb{C}^2$, $(0, 0) \in \Gamma$, determined implicitly by the equation*

$$\eta^2 = a\xi + d\mathbb{I}(\xi, \eta) + 2e\xi\eta + \partial_1 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta), \tag{22}$$

with $\mathbb{I}(0, 0) = 0$ and such that, for any $\zeta = (\xi, \eta) \in \Gamma \cap \mathbb{R}^2$, the two-dimensional torus

$$\mathcal{T}^{(0)}_{\xi,\eta} = \{(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} : q = \xi,\ J_1 = \mathbb{I}(\xi, \eta),\ J_2 = 2\xi\eta,\ p = 0\}$$

is invariant under the flow of $\breve{\mathcal{Z}}$ with parallel dynamics for ϕ determined by the vector $\Omega = (\Omega_1, \Omega_2)$ of intrinsic frequencies:

$$\Omega_1(\xi, \eta) = \omega_1 + b\mathbb{I}(\xi, \eta) + d\xi + 2f\xi\eta + \partial_2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_1}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}, \tag{23}$$

$$\Omega_2(\xi, \eta) = \omega_2 + \eta + 2c\xi\eta + e\xi + f\mathbb{I}(\xi, \eta) + \tfrac{1}{2}\partial_3 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_2}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}. \tag{24}$$

Moreover, for $\xi > 0$, the corresponding tori of $\mathcal{Z}$ are real.

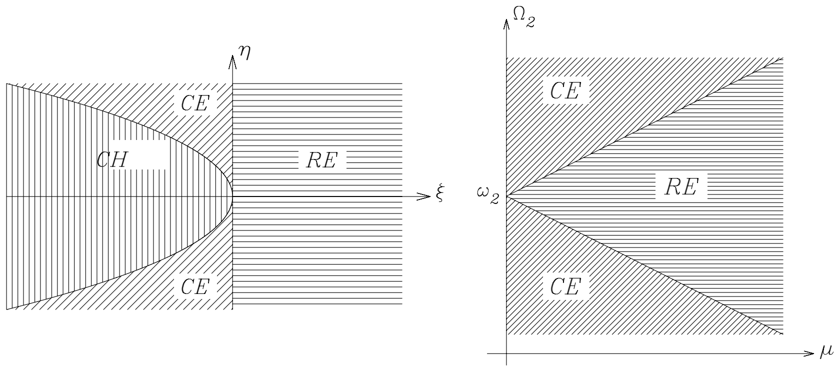

Figure 1. Qualitative plots of the distribution of invariant tori of the normal form, linked to the direct periodic Hopf bifurcation, in the parameter spaces (ξ, η) and (μ, Ω_2). The acronyms R, C, E and H indicate real, complex, elliptic and hyperbolic tori, respectively. In the left plot, the curve separating CE and CH (which is close to the parabola $\xi = -2\eta^2/a$) corresponds to complex parabolic tori, whilst the line $\xi = 0$ and the curves separating RE and CE in the right plot (which are close to the straight lines $\Omega_2 = \omega_2 \pm \mu/2$) correspond to stable periodic orbits.

Remark 4.2. If we set $\xi = 0$, then $\mathcal{T}^{(0)}_{0,\eta}$ corresponds to the family of periodic orbits of $\mathcal{Z}$ in which the critical one is embedded, but only those on the stable side of the bifurcation. These periodic orbits are parametrized by $q = p = J_2 = 0$ and $J_1 = \mathbb{I}(0, \eta) =: \tilde{\mathbb{I}}(\eta^2)$, and hence the periodic orbit given by η is the same given by $-\eta$. The angular frequency of the periodic orbit $\mathcal{T}^{(0)}_{0,\eta}$ is given by $\Omega_1(0, \eta) =: \tilde{\Omega}_1(\eta^2)$ and the two normal ones are $\Omega_2(0, \eta) =: \eta + \tilde{\Omega}_2(\eta^2)$ and $-\eta + \tilde{\Omega}_2(\eta^2)$ (check it in the Hamiltonian equations of (15)). We observe that $\Omega_2(0, \eta)$ depends on the sign of η, but that to change η by $-\eta$ only switches both normal frequencies. Moreover, the functions $\tilde{\mathbb{I}}$, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are analytic around the origin and, as a consequence of the normal form criteria of theorem 4.1, they are independent of R and give the parametrization of the family of periodic orbits of (14) and of their intrinsic and normal frequencies. See figure 1.

The proof of theorem 4.2 follows directly by substitution in the Hamilton equations of $\breve{\mathcal{Z}}$. Here we shall only stress that $d \neq 0$ is the only necessary hypothesis for the implicit function $\mathbb{I}$ to exist in a neighbourhood of (0, 0). In its turn, the reality condition follows at once writing the invariant tori in the former coordinates (θ, x, I, y) (see (19)). Explicitly, the corresponding quasiperiodic solutions are

$$\theta = \Omega_1(\zeta)t + \phi_1^{(0)}, \qquad x_1 = \sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}), \qquad x_2 = -\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}),$$
$$I = \mathbb{I}(\zeta), \qquad y_1 = -\eta\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}), \qquad y_2 = -\eta\sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}).$$

Therefore, $\zeta = (\xi, \eta)$ are the parameters of the family of tori, so they 'label' an specific invariant torus of $\mathcal{Z}$. Classically, when applying KAM methods, it is usual to require the *frequency map*, $\zeta \mapsto \Omega(\zeta)$, to be a diffeomorphism, so that we can label the tori in terms of its vector of intrinsic frequencies. This is (locally) achieved by means of the standard Kolmogorov non-degeneracy condition, $\det(\partial_\zeta \Omega) \neq 0$. In the present case, simple computations show that

$$\mathbb{I}(\xi, \eta) = -\frac{a}{d}\xi + \cdots, \qquad \Omega_1(\xi, \eta) = \omega_1 + \left(d - \frac{ab}{d}\right)\xi + \cdots,$$
$$\Omega_2(\xi, \eta) = \omega_2 + \left(e - \frac{af}{d}\right)\xi + \eta + \cdots \tag{25}$$

(for higher order terms see [39]). Then, Kolmogorov's condition computed at the resonant orbit reads as $d - ab/d \neq 0$. Although this is the classic approach, we shall be forced to choose a set of parameters on the family different from the intrinsic frequencies.

Remark 4.3. The Lagrangian 3D tori of the normal form $\breve{\mathcal{Z}}$ are given in terms of the periodic solutions of the *reduced* one degree of freedom Hamiltonian (see [39])

$$\breve{\mathcal{Z}}'(q, p; J_1, J_2) = qp^2 + \frac{J_2^2}{4q} + (dJ_1 + eJ_2)q + \frac{a}{2}q^2 + Z'(q; J_1, J_2),$$

with $Z'(q; J_1, J_2) = Z(q, J_1, J_2/2) - Z(0, J_1, J_2/2)$, where the (first integrals) J_1 and J_2 have to be dealt with as parameters. If we set $E' = \breve{\mathcal{Z}}'(q, p; J_1, J_2)$, then the differential equation for q is

$$\ddot{q} = 2E' - 4(dJ_1 + eJ_2)q - 3aq^2 - 2(Z' + q\partial_1 Z').$$

This means that, if we get rid of the contribution of Z', then 'first order' parametrizations of such periodic orbits can be given in terms of *Weierstraß elliptic functions*.

4.3. Normal behaviour of the bifurcated tori

Let us consider the variational equations of $\breve{\mathcal{Z}}$ around the family of (real) bifurcated tori $\mathcal{T}^{(0)}_{\xi,\eta}$ (with $\xi > 0$). The restriction of these equations to the normal directions (q, p) is given by a two-dimensional linear system with constant coefficients, with matrix

$$M_{\xi,\eta} = \begin{pmatrix} 0 & 2\xi \\ -\dfrac{2\eta^2}{\xi} - a - \partial^2_{1,1} Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) & 0 \end{pmatrix}. \qquad (26)$$

Then, the *characteristic exponents* (or *normal eigenvalues*) of this torus are

$$\lambda_\pm(\xi, \eta) = \pm\sqrt{-4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1} Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta)}. \qquad (27)$$

If $a > 0$, it is easy to realize that the eigenvalues $\lambda_\pm$ are purely imaginary if $\xi > 0$ and η are both small enough, and hence the family $\mathcal{T}^{(0)}_{\xi,\eta}$ holds only elliptic tori. If $a < 0$, then elliptic, hyperbolic and parabolic tori co-exist simultaneously in the family. In this paper we are only interested in the case $a > 0$ (direct bifurcation), so from now on we shall only be concerned with elliptic tori. Hence, we denote by $\mu = \mu(\xi, \eta) > 0$ the only *normal frequency* of the torus $\mathcal{T}^{(0)}_{\xi,\eta}$, so that $\lambda_\pm = \pm \mathrm{i}\mu$, with

$$\mu^2 := 2\xi\partial^2_{q,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta} = 4\eta^2 + 2a\xi + 2\xi\partial^2_{1,1} Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta). \qquad (28)$$

If we pick up the (stable) periodic orbit $(0, \pm\eta)$, then $\mu = 2|\eta|$. Hence, it is clear that $\mu \to 0$ as we approach the resonant orbit $\xi = \eta = 0$. Thus, the elliptic bifurcated tori of the normal form are very close to parabolic. This is the main source of problems when proving their persistence in the complete system.

Remark 4.4. Besides those having $0 < \xi \ll 1$ and $|\eta| \ll 1$ we observe that, from formula (27), those tori having $\xi < 0$ but $4\eta^2 + 2a\xi + 2\xi\partial^2_{1,1} Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) > 0$ are elliptic too, albeit they are complex tori when written in the original variables (recall that $\xi = 0$ corresponds to the stable periodic orbits of the family, see remark 4.2). However, when performing the KAM scheme, we will work with them all together (real or complex tori), because they turn to be real when written in the 'action–angle' variables introduced in (19). The discussion between real or complex tori of the original system (10) is carried on in section 5.13. See figure 1.

5. Proof of theorem 3.1

We consider the initial Hamiltonian $\mathcal{H}$ in (10) and take $R > 0$, small enough, fixed from now on. Then we compute the normal form of $\mathcal{H}$ up to a 'suitable order', depending on R, as stated in theorem 4.1. As the normalizing transformation $\hat{\Psi}^{(R)}$ depends on the selected R, it is clear that the transformed Hamiltonian $\mathcal{H} \circ \hat{\Psi}^{(R)}$ also does. However, as R is fixed, in the following we drop the explicit dependence on R unless it is strictly necessary. Now we introduce the canonical coordinates (19) and obtain the Hamiltonian $\check{\mathcal{H}}$ in (20). Then the keystone of the proof of theorem 3.1 is a KAM process applied to $\check{\mathcal{H}}$.

To carry out this procedure, first in section 5.1 we discuss which is the vector Λ of basic frequencies we use to label the 2D-bifurcated tori. In section 5.2 we introduce Λ as a parameter on the Hamiltonian $\check{\mathcal{H}}$. Moreover, the resulting system is complexified in order to simplify the resolution of the homological equations. The iterative KAM scheme we perform is explained in section 5.3. We also discuss the main difficulties we found when applying this process—in the present close-to-parabolic setting—with respect to the standard non-degenerate context. To justify the validity of our approach, the particular non-degeneracy condition linked to this construction is checked in section 5.4. In section 5.5 we explain how we carry out in the KAM process the ultra-violet cut-off with respect to the angles of the tori. This cut-off is performed in order to prove the Whitney-smoothness, with respect to the parameter Λ, of the surviving tori. After that, we begin with the quantitative part of the proof. To do that, first we have to select the initial set of basic frequencies in which we look for the corresponding invariant torus (section 5.6). Then, we have to control the bounds on the initial family of Hamiltonians (section 5.7), the quantitative estimates on the KAM iterative process introduced before (section 5.8) and the convergence of this procedure in a suitable set of basic frequencies (sections 5.9 and 5.10). To discuss the measure of this set we use Lipschitz constants. In section 5.11 we assure that we have a suitable control on these constants, whilst in section 5.12 we properly control this measure. Finally, in section 5.13 we discuss which of the invariant tori we have obtained are real when expressed in the original coordinates and, in section 5.14, we establish the Whitney-C^∞ smoothness of the family.

5.1. Lack of parameters

One of the problems intrinsically linked to the perturbation of elliptic invariant tori is the so-called 'lack of parameters'. In fact, this is a common difficulty in the theory of quasiperiodic motions in dynamical systems (see [11,37,51]). Basically, it implies that one cannot construct a perturbed torus with a fixed set of (Diophantine) intrinsic and normal frequencies, for the system does not contain enough internal parameters to control them all simultaneously. All that one can expect is to build perturbed tori with only a given subset of *basic frequencies* previously fixed (equal to the numbers of parameters that one has). The remaining frequencies have to be dealt with (when possible) as a function of the prefixed ones. As explained in the introduction, a different approach is to use parametrized KAM theory (see references quoted there).

Let us suppose for the moment that, in our case, the two intrinsic frequencies could be the basic ones and that the normal frequency is a function of the intrinsic ones (this is the standard approach). These three frequencies are present on the (small) denominators of the KAM iterative scheme (see (29)). It means that to carry out the first step of this process, one has to restrict the parameter set to the intrinsic frequencies so that they, together with the corresponding normal one of the unperturbed torus, satisfy the required Diophantine conditions (see (69)). After this first step, we can only keep fixed the values of the intrinsic frequencies (assuming Kolmogorov non-degeneracy), but the function giving the normal frequency of the new approximation to the invariant torus has changed. Thus, we cannot guarantee *a priori* that the new normal frequency is non-resonant with the former intrinsic ones.

To succeed in the iterative application of the KAM process, it is usual to ask for the denominators corresponding to the unperturbed tori to move when the basic frequencies do. In our context, with only one frequency to control, this is guaranteed if we can add suitable non-degeneracy conditions on the function giving the normal frequency. These *transversality conditions* avoid the possibility that one of the denominators falls permanently inside a resonance and allows one to obtain estimates for the Lebesgue measure of the set of 'good' basic frequencies at any step of the iterative process. For 2D-elliptic low-dimensional tori with only one normal frequency, the denominators to be taken into account are[4] (the so-called *Mel'nikov's second non-resonance condition*, see [35, 36])

$$\mathrm{i}\langle k, \Omega\rangle + \mathrm{i}\ell\mu, \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \quad \forall \ell \in \{0, \pm 1, \pm 2\}, \tag{29}$$

where $\Omega \in \mathbb{R}^2$ are the intrinsic frequencies and $\mu = \mu(\Omega) > 0$ the normal one. Now we compute the gradient with respect to Ω of such divisors and require them not to vanish. These transversality conditions are equivalent to $2\nabla_\Omega \mu(\Omega) \notin \mathbb{Z}^2 \setminus \{0\}$. For equivalent conditions in the 'general' case see [28]. For weak non-degeneracy conditions see [11, 49, 51–53].

This, however, does not work in the current situation. To realize, a glance at (25) shows that the first order expansion, at $\Omega = \omega$, of the inverse of the frequency map is

$$\xi = \frac{d}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots, \qquad \eta = \frac{af - ed}{d^2 - ab}(\Omega_1 - \omega_1) + \Omega_2 - \omega_2 + \cdots.$$

Now, substitution in expression (27) gives for the normal frequency

$$\mu(\Omega) = \sqrt{\frac{2ad}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots},$$

so $\nabla_\Omega \mu(\Omega)$ is not well defined at the critical periodic orbit. Therefore, we use different parameters on the family. From (25) and (27), it can be seen that ξ and η may be expressed as a function of μ and Ω_2,

$$\xi = \frac{\mu^2}{2a} - \frac{2}{a}(\Omega_2 - \omega_2)^2 + \cdots,$$
$$\eta = \Omega_2 - \omega_2 + \left(\frac{f}{2d} - \frac{e}{2a}\right)\mu^2 + \left(\frac{2e}{a} - \frac{3f}{d}\right)(\Omega_2 - \omega_2)^2 + \cdots.$$

Now, let us denote $\Lambda = (\mu, \Omega_2)$ the new set of basic frequencies and write Ω_1 as a function of them. Substitution in the expression for Ω_1 in (25) yields

$$\Omega_1^{(0)}(\mu, \Omega_2) := \omega_1 + \left(\frac{d}{2a} - \frac{b}{2d}\right)\mu^2 + \left(\frac{3b}{d} - \frac{2d}{a}\right)(\Omega_2 - \omega_2)^2 + \cdots. \tag{30}$$

The derivatives with respect to Λ of the KAM denominators (29) are, at the critical periodic orbit,

$$\nabla_\Lambda \left(k_1\Omega_1^{(0)}(\Lambda) + k_2\Omega_2 + \ell\mu\right)\Big|_{\Lambda=(0,\omega_2)} = (\ell, k_2), \qquad k_1, k_2, \ell \in \mathbb{Z}, \text{ with } |\ell| \leqslant 2. \tag{31}$$

So the divisors will change with Λ whenever the integer vector $(\ell, k_2) \neq (0, 0)$. But if $\ell = k_2 = 0$ then $k_1 \neq 0$, and the modulus of the divisor $k_1\Omega_1^{(0)}(\Lambda)$ will be bounded from below.

[4] Bourgain showed in [5, 6] that, in order to prove the existence of these tori, conditions with $\ell = \pm 2$ can be omitted. However, the proof becomes extremely involved and the linear normal behaviour of the obtained tori cannot be controlled.

Remark 5.1. As we have already mentioned in the introduction, there is also the possibility of using the frequency ratio $[\mu : \Omega_1 : \Omega_2]$ as a parameter of the tori. In particular, the choices $\Lambda = (\mu/\Omega_1, \Omega_2/\Omega_1)$ or $\Lambda = (\mu/\Omega_2, \Omega_1/\Omega_2)$ would work as well as our selection of basic frequencies, since the gradient of the KAM denominators (29) with respect to any of them produces similar expressions as in (31), giving rise to the same transversality condition $(\ell, k_2) \neq (0, 0)$. Nevertheless, our choice of basic frequencies is more suitable for our purposes in order to have homological equations as simple as possible.

5.2. Expansion around the unperturbed tori and complexification of the system

Once we have selected the parameters on the family, the next step is to put system (20) into a more suitable form. Concretely, we replace the Hamiltonian $\breve{\mathcal{H}}$ by a family of Hamiltonians, $H_\Lambda^{(0)}$, having as a parameter the vector of basic frequencies Λ. This is done by placing 'at the origin' the invariant torus of the 'unperturbed Hamiltonian' $\breve{\mathcal{Z}}$, corresponding to the parameter Λ, and then arranging the corresponding normal variational equations of $\breve{\mathcal{Z}}$ to the diagonal form and uncoupling (up to first order) the 'central' and normal terms around the torus. This means to remove the quadratic term $[\cdot]_{I,z}$ (see (4)) from the unperturbed part of (34).

If for the moment we set the perturbation $\breve{\mathcal{R}}$ to zero, then $H_\Lambda^{(0)}$ constitutes a family of analytic Hamiltonians so that, for a given $\Lambda = (\mu, \Omega_2)$, the corresponding member has at the origin a 2D-elliptic invariant torus with normal frequency μ and intrinsic frequencies $(\Omega_1^{(0)}(\Lambda), \Omega_2)$, where $\Omega_1^{(0)}(\Lambda)$ is defined through (30). Our target is to prove that if we take the perturbation $\breve{\mathcal{R}}$ into account then, for most of the values of Λ (in a Cantor set), the full system $H_\Lambda^{(0)}$ has an invariant 2D-elliptic torus close to the origin, with the same vector of basic frequencies Λ, but perhaps with a different Ω_1. Similar ideas have been used in [28, 29].

To introduce $H_\Lambda^{(0)}$ we consider the family of symplectic transformations $(\theta_1, \theta_2, x, I_1, I_2, y) \mapsto (\phi_1, \phi_2, q, J_1, J_2, p)$, defined for $\Lambda \in \Gamma$ (see theorem 4.2) and given by

$$
\begin{aligned}
\phi_1 &= \theta_1 - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right), & J_1 &= \mathbb{I}(\zeta) + I_1,\\
\phi_2 &= \theta_2 - \frac{2\xi}{\mu^2}(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right), & J_2 &= 2\xi\eta + I_2, \qquad (32)\\
q &= \xi + x - \frac{\xi}{\lambda_+}y - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})I_1 - \frac{2\xi}{\mu^2}(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})I_2, & p &= \frac{\lambda_+}{2\xi}x + \frac{1}{2}y,
\end{aligned}
$$

where $\lambda_+ = \mathrm{i}\mu$. Although it has not been written explicitly, the parameters $\zeta = (\xi, \eta)$ must be thought of as functions of the basic frequencies Λ, i.e. $\zeta = \zeta(\Lambda)$.

This transformation can be read as the composition of two changes. One is the symplectic 'diagonalizing' change

$$
Q = x - \frac{\xi}{\lambda_+}y, \qquad P = \frac{\lambda_+}{2\xi}x + \frac{1}{2}y, \qquad (33)
$$

which puts the normal variational equations—associated with the unperturbed part $\breve{\mathcal{Z}}$—of the torus into the diagonal form. We point out that we choose (33) as a diagonalizing change because it skips any square root of ξ or μ. The other change moves the torus to the origin and gets rid of the contribution of $\breve{\mathcal{Z}}$ to the term $[\cdot]_{I,z}$ of the Taylor expansion of $H_\Lambda^{(0)}$ (recall (28)). To diagonalize the normal variationals and to kill this coupling term is not strictly necessary, but both operations simplify a lot the homological equations of the KAM process (see (eq1)–(eq5)).

Note that the linear change (33) is a complexification of the real Hamiltonian (20), i.e. the *real* values of the normal variables (q, p) correspond now to *complex* values of (x, y). Nevertheless, the invariant tori of (34) we finally obtain are real tori when expressed in coordinates $(\phi_1, \phi_2, q, J_1, J_2, p)$ and those having $q > 0$ are also real in the original variables (through change (19)). The real character of the tori of (34) can be verified in two ways. The first one is to overcome the complexification (33) and to perform the KAM process by using the real variables (Q, P) instead of (x, y). The price we paid for using this methodology is that the solvability of the homological equations—of the iterative KAM process—becomes more involved, because they are no longer diagonal. The other way to proceed is to observe that the complexified homological equations have a unique (complex) solution. Thus, as we are dealing with linear (differential) equations, it implies that the corresponding real homological equations, written in terms of the variables (Q, P), also have a unique (real) solution. Hence, as the complexification (33) is canonical, it means that if we express the generating function S (see (40)) obtained as a solution of the homological equations in the real variables (Q, P), then we obtain a real generating function (see remark 5.2 for more details). Consequently, the symmetries introduced by the complexification are kept after any step of the iterative KAM process, and we can go back to a real Hamiltonian by means of the inverse transformation of (33). Thus, for simplicity, we have preferred to follow this second approach and to use complex variables.

In this way, the Hamiltonian $\breve{\mathcal{H}}$ in (20) casts into $H^{(0)}_{\Lambda} = H^{(0)}_{\Lambda}(\theta, x, I, y)$, with

$$H^{(0)}_{\Lambda} = \phi^{(0)}(\Lambda) + \langle \Omega^{(0)}(\Lambda), I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}(\Lambda) z\rangle + \tfrac{1}{2}\langle I, \mathcal{C}^{(0)}(\Lambda) I\rangle + \tilde{H}^{(0)}(x, I, y; \Lambda) + \hat{H}^{(0)}(\theta, x, I, y; \Lambda). \tag{34}$$

Here, $\tilde{H}^{(0)}$ holds the terms of order greater than two in (z, I), where $z = (x, y)$, coming from the normal form $\breve{\mathcal{Z}}$, i.e. $[\tilde{H}^{(0)}] = \tilde{H}^{(0)}$ (see (4)), and $\hat{H}^{(0)}$ is the transform of the remainder $\breve{\mathcal{R}}$, whereas

$$\phi^{(0)}(\Lambda) = \breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}}, \qquad \Omega^{(0)}_1(\Lambda) = \partial_{J_1}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}}, \qquad \Omega^{(0)}_2(\Lambda) = \Omega_2, \qquad \mathcal{B}(\Lambda) = \begin{pmatrix} 0 & \lambda_+ \\ \lambda_+ & 0 \end{pmatrix}, \tag{35}$$

(see (23), (24), (26)–(28) and (30)) and the symmetric matrix $\mathcal{C}^{(0)}$ is given by

$$\begin{aligned} \mathcal{C}^{(0)}_{1,1}(\Lambda) &= \partial^2_{J_1,J_1}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}} - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}})^2 = b + \partial^2_{2,2}Z - \frac{2\xi}{\mu^2}(d + \partial^2_{1,2}Z)^2, \\ \mathcal{C}^{(0)}_{1,2}(\Lambda) &= \partial^2_{J_1,J_2}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}} - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}})(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}}) \\ &= f + \frac{1}{2}\partial^2_{2,3}Z - \frac{2\xi}{\mu^2}(d + \partial^2_{1,2}Z)\left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial^2_{1,3}Z\right), \\ \mathcal{C}^{(0)}_{2,2}(\Lambda) &= \partial^2_{J_2,J_2}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}} - \frac{2\xi}{\mu^2}(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}})^2 = \frac{1}{2\xi} + c + \frac{1}{4}\partial^2_{3,3}Z - \frac{2\xi}{\mu^2}\left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial^2_{1,3}Z\right)^2, \end{aligned} \tag{36}$$

where the partial derivatives of Z given above are evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$. If the remainder $\hat{H}^{(0)}$ is not taken into account, then $I = 0$, $z = 0$ corresponds to an invariant 2D-elliptic torus of $H^{(0)}_{\Lambda}$ with basic frequency vector Λ. The normal variational equations of this torus are given by the (complex) diagonal matrix $\mathcal{J}_1\mathcal{B}$.

5.3. The iterative scheme

Now we proceed to describe (here only formally) the KAM iterative procedure we use to construct the elliptic two-dimensional tori. The underlying idea goes back to Kolmogorov in [32] and Arnol'd in [2, 3]. In what concerns low-dimensional tori, see references quoted in the introduction.

We perform a sequence of canonical changes on $H_\Lambda^{(0)}$ (see (34)), depending on the parameter Λ, thus obtaining a sequence of Hamiltonians $\{H_\Lambda^{(n)}\}_{n\geqslant 0}$, with a limit Hamiltonian $H_\Lambda^{(\infty)}$ having at the origin a 2D-elliptic invariant torus, with $\Lambda = (\mu, \Omega_2)$ as a vector of basic frequencies. Concretely, we want $H_\Lambda^{(\infty)}$ to be of the form

$$H_\Lambda^{(\infty)}(\theta, x, I, y) = \phi^{(\infty)}(\Lambda) + \langle \Omega^{(\infty)}(\Lambda), I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}(\Lambda)z\rangle \\ +\tfrac{1}{2}\langle I, \mathcal{C}^{(\infty)}(\theta;\Lambda)I\rangle + \tilde{H}^{(\infty)}(\theta, x, I, y;\Lambda), \tag{37}$$

with $[\tilde{H}^{(\infty)}] = \tilde{H}^{(\infty)}$, the matrix $\mathcal{B}$ given by (35) and the function $\Omega^{(\infty)}(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$. This process is built as a Newton-like iterative method, yielding 'quadratic convergence' if we restrict to the values of Λ for which suitable Diophantine conditions hold at any step. We point out that, albeit $\mathcal{C}^{(0)}$ and $\tilde{H}^{(0)}$ are independent of θ, this property is not kept by the iterative process.

To describe a generic step of this iterative scheme we consider a Hamiltonian of the form (see (4))

$$H = \mathrm{a}(\theta) + \langle \mathrm{b}(\theta), z\rangle + \langle \mathrm{c}(\theta), I\rangle + \tfrac{1}{2}\langle z, \mathrm{B}(\theta)z\rangle + \langle I, \mathrm{E}(\theta)z\rangle + \tfrac{1}{2}\langle I, \mathrm{C}(\theta)I\rangle + \Xi(\theta, x, I, y). \tag{38}$$

Although we do not write this dependence explicitly, we suppose that H also depends on Λ (recall that everything also depends on the prefixed R). Moreover, we also assume that if we replace the 'complex' variables (x, y) by (Q, P) through (33), then H becomes a real analytic function. For any $\Lambda = (\mu, \Omega_2)$ we define from (38)

$$\bar{H} = \langle \mathrm{a}\rangle_\theta + \langle \Omega, I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}z\rangle + \tfrac{1}{2}\langle I, \mathrm{C}(\theta)I\rangle + \Xi(\theta, x, I, y), \tag{39}$$

and suppose that $H - \bar{H}$ is 'small'. To fix ideas, assume $H - \tilde{H} = \mathrm{O}(\varepsilon)$ with ε decreasing to zero along the steps of the iterative process. We point out that if we start the iterative process with $H^{(0)}$ in (34), then $\varepsilon = \mathrm{O}(\hat{H}^{(0)})$. The Hamiltonian $\bar{H}$ looks like (37), which is the form we want for the limit Hamiltonian, with $\Omega = (\Omega_1, \Omega_2)$, for certain $\Omega_1 = \Omega_1(\Lambda)$ to be chosen iteratively (initially we take $\Omega_1 = \Omega_1^{(0)}$ of (30)), and $\mathcal{B}(\Lambda)$ defined by (35) is held fixed during the iterative process. Moreover, we also assume that the matrix C is close to $\mathcal{C}^{(0)}(\Lambda)$ defined by (36), but we do not require C to remain constant with the step.

Now we perform a canonical change on H so that it squares the size of ε. Concretely, if we call $H^{(1)}$ the transformed Hamiltonian, expand $H^{(1)}$ as H in (38) and define $\bar{H}^{(1)}$ from $H^{(1)}$ as in (39), we want (roughly speaking) the norm of $H^{(1)} - \bar{H}^{(1)}$ to be of $\mathrm{O}(\varepsilon^2)$.

The canonical transformations we use are defined by the time-one flow of a suitable Hamiltonian $S = S_\Lambda$, the so-called *generating function* of the change, which we denote as $\Psi_{t=1}^S$ or simply Ψ_1^S (see section 2). Precisely, we look for S of the form (compare [4, 28, 29])

$$S(\theta, x, I, y) = \langle \chi, \theta\rangle + \mathrm{d}(\theta) + \langle \mathrm{e}(\theta), z\rangle + \langle \mathrm{f}(\theta), I\rangle + \tfrac{1}{2}\langle z, \mathrm{G}(\theta)z\rangle + \langle I, \mathrm{F}(\theta)z\rangle, \tag{40}$$

where $\chi \in \mathbb{C}^2$, $\langle \mathrm{d}\rangle_\theta = 0$, $\langle \mathrm{f}\rangle_\theta = 0$ and G is a symmetric matrix with $\langle \mathrm{G}_{1,2}\rangle_\theta = \langle \mathrm{G}_{2,1}\rangle_\theta = 0$.

Remark 5.2. The above conditions guarantee the uniqueness of S as a solution of the homological equations (eq1)–(eq5). Furthermore, as we want to ensure that we have a real generating function after applying the inverse of (33) to S, we have to require that $\chi \in \mathbb{R}^2$ and that $\mathrm{d}(\theta)$, $\mathcal{S}^*\mathrm{e}(\theta)$, $f(\theta)$, $\mathcal{S}^*\mathrm{G}(\theta)\mathcal{S}$ and $\mathrm{F}(\theta)\mathcal{S}$ are real functions, where $\mathcal{S}$ is the matrix of the inverse of the linear change (33). So, if we set $\mathcal{G}(\theta) = \mathcal{S}^*\mathrm{G}(\theta)\mathcal{S}$, condition $\langle \mathrm{G}_{1,2}\rangle_\theta = 0$ reads, for the real matrix $\mathcal{G}$, $4\xi^2\langle \mathcal{G}_{1,1}\rangle_\theta + \mu^2\langle \mathcal{G}_{2,2}\rangle_\theta = 0$. If we assume that these $\mathcal{S}$-symmetries hold for H, then it is clear that they also hold for S.

Then we have

$$H^{(1)} := H \circ \Psi_1^S = H + \{H, S\} + \int_0^1 (1-t)\{\{H, S\}, S\} \circ \Psi_t^S \, \mathrm{d}t.$$

By assuming *a priori* that S is small, of $\mathrm{O}(\varepsilon)$, we select S so that $H + \{\bar{H}, S\}$ takes the form

$$H + \{\bar{H}, S\} = \phi^{(1)} + \langle \Omega^{(1)}, I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}z\rangle + \tfrac{1}{2}\langle I, \mathcal{C}^{(1)}(\theta) I\rangle + \tilde{H}^{(1)}(\theta, x, I, y),$$

being $\Omega^{(1)} = (\Omega_1^{(1)}, \Omega_2)$ with $\tilde{H}^{(1)}$ holding the terms of higher degree, i.e. $\tilde{H}^{(1)} = [H+\{\bar{H}, S\}]$. If we write these conditions in terms of H and the generating function S, this leads to the following *homological equations* (see (8)):

$$\{\mathrm{a}\}_\theta - L_\Omega \mathrm{d} = 0, \tag{eq1}$$

$$\mathrm{b} - L_\Omega \mathrm{e} + \mathcal{B}\mathcal{J}_1 \mathrm{e} = 0, \tag{eq2}$$

$$c - \Omega^{(1)} - L_\Omega \mathrm{f} - \mathrm{C}\left(\chi + (\partial_\theta \mathrm{d})^*\right) = 0, \tag{eq3}$$

$$\tilde{\mathrm{B}} - \mathcal{B} - L_\Omega \mathrm{G} + \mathcal{B}\mathcal{J}_1 \mathrm{G} - \mathrm{G}\mathcal{J}_1\mathcal{B} = 0, \tag{eq4}$$

$$\tilde{\mathrm{E}} - L_\Omega \mathrm{F} - \mathrm{F}\mathcal{J}_1\mathcal{B} = 0, \tag{eq5}$$

where

$$\Omega_1^{(1)} := \langle c_1\rangle_\theta - \left\langle \mathrm{C}_{1,1}\left(\chi_1 + \partial_{\theta_1}\mathrm{d}\right)\right\rangle_\theta - \left\langle \mathrm{C}_{1,2}\left(\chi_2 + \partial_{\theta_2}\mathrm{d}\right)\right\rangle_\theta, \tag{41}$$

$$\tilde{\mathrm{B}} := \mathrm{B} - \left[\partial_I \Xi\left(\chi + (\partial_\theta \mathrm{d})^*\right) - \partial_z \Xi \mathcal{J}_1 \mathrm{e}\right]_{(z,z)}, \tag{42}$$

$$\tilde{\mathrm{E}} := \mathrm{E} - \mathrm{C}\,(\partial_\theta \mathrm{e})^* - \left[\partial_I \Xi\left(\chi + (\partial_\theta \mathrm{d})^*\right) - \partial_z \Xi \mathcal{J}_1 \mathrm{e}\right]_{(I,z)}. \tag{43}$$

Prior to solving these equations completely, we want to discuss the reason for the definition of $\Omega_1^{(1)}$ and how the constant vector χ is fixed, because these are the most involved issues when solving them. These quantities are used to adjust the average of some components of the homological equations, ensuring the compatibility of the full system when they are appropriately chosen. First, $\Omega_1^{(1)}$ is defined so that the average of the first component of the (vectorial) equation (eq3) is zero. Moreover, as one wants Ω_2 and μ not to change from one iterate to another, χ must satisfy the linear system formed by the second component of (eq3) and the first row second column component of the (matricial) equation (eq4) (or, by symmetry, the second row first column of this equation). One obtains the linear system

$$\langle \mathcal{A}\rangle_\theta \chi = -\mathrm{h}, \tag{44}$$

where

$$\mathcal{A}(\theta) = \begin{pmatrix} \mathrm{C}_{2,1}(\theta) & \mathrm{C}_{2,2}(\theta) \\ \partial^3_{I_1,x,y}\Xi(\theta, 0) & \partial^3_{I_2,x,y}\Xi(\theta, 0) \end{pmatrix} \tag{45}$$

and the components of the right-hand term in (44) are

$$\mathrm{h}_1 := \Omega_2 - \langle \mathrm{c}_2\rangle_\theta + \left\langle \mathrm{C}_{2,1}\partial_{\theta_1}\mathrm{d}\right\rangle_\theta + \left\langle \mathrm{C}_{2,2}\partial_{\theta_2}\mathrm{d}\right\rangle_\theta, \tag{46}$$

$$\begin{aligned}\mathrm{h}_2 := \lambda_+ - \left\langle \mathrm{B}_{1,2}\right\rangle_\theta + \left\langle \partial^3_{I_1,x,y}\Xi(\theta,0)\partial_{\theta_1}\mathrm{d}\right\rangle_\theta + \left\langle \partial^3_{I_2,x,y}\Xi(\theta,0)\partial_{\theta_2}\mathrm{d}\right\rangle_\theta \\ + \left\langle \partial^3_{x,y,y}\Xi(\theta,0)\mathrm{e}_1\right\rangle_\theta - \left\langle \partial^3_{x,x,y}\Xi(\theta,0)\mathrm{e}_2\right\rangle_\theta.\end{aligned} \tag{47}$$

Hence, to ensure the compatibility of the homological equations, it is necessary to see that the matrix $\langle \mathcal{A}\rangle_\theta$ is not singular and (in order to bound the solutions of the system (44) later on) to derive suitable estimates for the norm of its inverse. This is the most important non-degeneracy condition to fulfil in order to ensure that we made a good selection of basic frequencies to label the tori. Thus, the next section is devoted to the verification of this condition for the unperturbed tori of $H^{(0)}$ (see (34)).

5.4. The non-degeneracy condition of the basic frequencies

Let us compute the matrix $\mathcal{A}$ associated with the 'unperturbed' terms of the Hamiltonian $H^{(0)}$, namely $\bar{\mathcal{A}}^{(0)}$. This matrix is defined by taking $\mathrm{C} = \mathcal{C}^{(0)}$ and $\Xi = \tilde{H}^{(0)}$ in (45) (see (34) and (36)). We observe that $\bar{\mathcal{A}}^{(0)}$ does not depend on θ, but this property is not kept for the matrices $\mathcal{A}$ of the iterative process. For $\tilde{H}^{(0)}$ we have (see (21) and (32))

$$\begin{aligned}
\partial^3_{I_1,x,y}\tilde{H}^{(0)}(0,0,0) &= -\frac{\xi}{\lambda_+}\partial^3_{J_1,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta} + \partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}\left(\frac{1}{\lambda_+} - \frac{2\xi^2}{\lambda_+^3}\partial^3_{q,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}\right) \\
&= -\frac{\xi}{\lambda_+}\partial^3_{1,1,2}Z + (d+\partial^2_{1,2}Z)\left(\frac{1}{\lambda_+} + 12\frac{\eta^2}{\lambda_+^3} - 2\frac{\xi^2}{\lambda_+^3}\partial^3_{1,1,1}Z\right), \\
\partial^3_{I_2,x,y}\tilde{H}^{(0)}(0,0,0) &= -\frac{\xi}{\lambda_+}\partial^3_{J_2,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta} + \partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}\left(\frac{1}{\lambda_+} - \frac{2\xi^2}{\lambda_+^3}\partial^3_{q,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta}\right) \\
&= -\frac{\xi}{\lambda_+}\left(2\frac{\eta}{\xi^2} + \frac{1}{2}\partial^3_{1,1,3}Z\right) + \left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial^2_{1,3}Z\right) \\
&\quad \times \left(\frac{1}{\lambda_+} + 12\frac{\eta^2}{\lambda_+^3} - 2\frac{\xi^2}{\lambda_+^3}\partial^3_{1,1,1}Z\right),
\end{aligned}$$

where the partial derivatives of Z are evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$, i.e. at the unperturbed torus. Then, simple (but tedious) computations show that

$$\det \bar{\mathcal{A}}^{(0)} = \mathcal{C}^{(0)}_{2,1}\partial^3_{I_2,x,y}\tilde{H}^{(0)}(0,0,0) - \mathcal{C}^{(0)}_{2,2}\partial^3_{I_1,x,y}\tilde{H}^{(0)}(0,0,0) = \frac{1}{\lambda_+^3}(\tilde{\mathcal{A}}^{(0)} + \hat{\mathcal{A}}^{(0)}),$$

where

$$\tilde{\mathcal{A}}^{(0)} = \frac{1}{\xi}\left(-(d+\partial^2_{1,2}Z)\left(\frac{\lambda_+^2}{2} + 2\eta^2\right) - \left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\eta(3\lambda_+^2 + 12\eta^2)\right),$$

$$\begin{aligned}
\hat{\mathcal{A}}^{(0)} &= (\xi\lambda_+^2\mathcal{C}^{(0)}_{2,2})\partial^3_{1,1,2}Z + \left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\left(2\xi\eta\partial^3_{1,1,1}Z - \frac{\xi\lambda_+^2}{2}\partial^3_{1,1,3}Z\right) \\
&\quad + (\lambda_+^2 + 12\eta^2 - 2\xi^2\partial^3_{1,1,1}Z)\left(\left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\left(e + \frac{1}{2}\partial^2_{1,3}Z\right) - \left(c + \frac{1}{4}\partial^2_{3,3}Z\right)(d + \partial^2_{1,2}Z)\right. \\
&\quad \left. + (d + \partial^2_{1,2}Z)\left(\xi\partial^3_{1,1,1}Z - 4\eta\left(e + \frac{1}{2}\partial^2_{1,3}Z\right) - \xi\partial^3_{1,1,3}Z\left(-\eta + e\xi + \frac{\xi}{2}\partial^2_{1,3}Z\right)\right)\right).
\end{aligned}$$

We remark that albeit $\mathcal{C}^{(0)}_{2,2}$ becomes singular when $\xi = \eta = 0$, the expression $\xi\lambda_+^2\mathcal{C}^{(0)}_{2,2}$ goes to zero when $\zeta = (\xi, \eta)$ does, and so does $\hat{\mathcal{A}}^{(0)}$. Now, taking into account definition (27), we replace

$$\lambda_+^2 = -4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1}Z$$

in the expression of $\tilde{\mathcal{A}}^{(0)}$. Then, some (nice) cancellations lead to the following expression:

$$\tilde{\mathcal{A}}^{(0)} = ad + d\partial^2_{1,1}Z + (6f\eta + 3\eta\partial^2_{2,3}Z + \partial^2_{1,2}Z)(a + \partial^2_{1,1}Z).$$

As a summary, we have that $\det \bar{\mathcal{A}}^{(0)} = (ad + \cdots)/\lambda_+^3$, where the terms denoted by dots vanish at the critical periodic orbit $\xi = \eta = 0$. Then, as $ad \neq 0$, we have for small values of ζ that $\det \bar{\mathcal{A}}^{(0)} \neq 0$. See section 5.7 for bounds on $(\bar{\mathcal{A}}^{(0)})^{-1}$.

5.5. The ultra-violet cut-off

Once we have fixed the way to compute χ, we discuss the solvability of the remaining part of the homological equations (eq1)–(eq5). By expanding them in Fourier series, we compute the different terms of S in (40) as solutions of small divisor equations. The divisors appearing are those specified in (29), which are integer combinations of the intrinsic frequencies $\Omega = (\Omega_1, \Omega_2)$ and of the normal one μ. For such divisors it is natural to ask for the following Diophantine conditions

$$|\langle k, \Omega\rangle + \ell\mu| \geqslant \tilde{\gamma}|k|_1^{-\tau}, \tag{48}$$

for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and $\ell \in \mathbb{Z}$, with $|\ell| \leqslant 2$, where $\tau > 1$ is the same as in (12) and $\tilde{\gamma} > 0$ (depending on R) will be specified later (see (69)). As Ω_1 will be dealt with as a function of Λ, we expect to have a Cantor set of values of Λ for which (48) holds. Moreover, as the function $\Omega_1 = \Omega_1(\Lambda)$ changes from one step to another, this Cantor set also changes (shrinks) with the step.

If at any step of the iterative scheme we restrict Λ to a Cantor set, then it is difficult to control the regularity with respect to Λ of the sequence of Hamiltonians $H^{(n)} = H^{(n)}_\Lambda$, because the parameter set has an empty interior. The Λ regularity is important, because it is used to control the (Lebesgue) measure of the 'bad' and 'good' parameters Λ along the iterative process (see section 5.12). For measure purposes, it is enough to use the Lipschitz dependence (see for instance [26–29]). In this work we have preferred to follow the approach of Arnol'd in [2, 3] and to deal with the analytic dependence with respect to Λ. This forces us to consider a KAM process with an *ultra-violet cut-off*. Concretely, we select a 'big' integer N, depending on the step and going to infinity, and consider the values of Λ for which (48) holds for any $k \in \mathbb{Z}^2 \setminus \{0\}$ and $|\ell| \leqslant 2$, but with $0 < |k|_1 < 2N$. This finite number of conditions defines a set with a non-empty interior for Λ that only becomes Cantor at the limit. Hence, the limit Hamiltonian is no longer analytic on Λ, but only C^∞ in the sense of Whitney (see appendix A.3).

Let us summarize the iterative scheme of section 5.3 and explain precisely how we introduced the ultra-violet cut-off. After N is fixed appropriately, we decompose the actual Hamiltonian H as (see (7))

$$H = H_{<N,\theta} + H_{\geqslant N,\theta}. \tag{49}$$

After that, $H_{<N,\theta}$ is arranged as H in (38) and then we apply the iterative scheme described in section 5.3 to $H_{<N,\theta}$ instead of H. This means that we compute the generating function $S = S_{<2N,\theta}$ in (40), by solving the homological equations (eq1)–(eq5) with $H_{<N,\theta}$ playing the rôle of H in (38), and hence, with $\bar{H}_{<N,\theta}$ playing the rôle of $\bar{H}$ in (39). The solution of such equations is given by (44) and by the following explicit formulae:

$$\mathrm{d} = \sum_{0<|k|_1<N} \frac{\mathrm{a}_k}{\mathrm{i}\langle k, \Omega\rangle} \exp\left(\mathrm{i}\langle k, \theta\rangle\right), \tag{50}$$

$$\mathrm{e}_j = \sum_{|k|_1<N} \frac{\mathrm{b}_{j;k}}{\mathrm{i}\langle k, \Omega\rangle + (-1)^{j+1}\lambda_+} \exp\left(\mathrm{i}\langle k, \theta\rangle\right), \tag{51}$$

$$\mathrm{f}_j = \sum_{0<|k|_1<2N} \frac{\tilde{\mathrm{c}}_{j;k}}{\mathrm{i}\langle k, \Omega\rangle} \exp\left(\mathrm{i}\langle k, \theta\rangle\right), \tag{52}$$

$$\{\mathrm{G}_{j,l}\}_\theta = \sum_{0<|k|_1<2N} \frac{\tilde{\mathrm{B}}_{j,l;k}}{\mathrm{i}\langle k, \Omega\rangle + 2(-1)^{j+1}\lambda_+\delta_{j,l}} \exp\left(\mathrm{i}\langle k, \theta\rangle\right), \tag{53}$$

$$\langle \mathrm{G}_{1,1}\rangle_\theta = \frac{\tilde{\mathrm{B}}_{1,1;0}}{2\lambda_+}, \qquad \langle \mathrm{G}_{2,2}\rangle_\theta = -\frac{\tilde{\mathrm{B}}_{2,2;0}}{2\lambda_+}, \tag{54}$$

$$F_{j,l} = \sum_{|k|_1<2N} \frac{\tilde{\mathrm{E}}_{j,l;k}}{\mathrm{i}\langle k, \Omega\rangle + (-1)^{l+1}\lambda_+} \exp\left(\mathrm{i}\langle k, \theta\rangle\right), \tag{55}$$

for $j, l = 1, 2$, where $\delta_{j,l}$ is Kronecker's delta and

$$\tilde{\mathrm{c}} = \{\mathrm{c}\}_\theta - \{C(\chi + (\partial_\theta \mathrm{d})^*)\}_\theta. \tag{56}$$

Then we can expand the new Hamiltonian $H^{(1)} := H \circ \Psi_1^S$ as

$$H^{(1)} = \phi^{(1)} + \langle \Omega^{(1)}, I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}z\rangle + \tfrac{1}{2}\langle I, \mathcal{C}^{(1)}(\theta) I\rangle + \tilde{H}^{(1)}(\theta, x, I, y) + \hat{H}^{(1)}(\theta, x, I, y),$$

where $\Omega_2^{(1)} = \Omega_2$ and $\Omega_1^{(1)}$ is given by (41). Moreover (see section 2 for notation)

$$\phi^{(1)} = \langle a\rangle_\theta - \langle \chi, \Omega\rangle, \qquad \mathcal{C}^{(1)} = C + [\{\bar{H}_{<N,\theta}, S\}]_{I,I}, \qquad \tilde{H}^{(1)} = \Xi + [\{\bar{H}_{<N,\theta}, S\}], \tag{57}$$

$$\hat{H}^{(1)} = \{H_{<N,\theta} - \bar{H}_{<N,\theta}, S\} + \int_0^1 (1-t)\{\{H_{<N,\theta}, S\}, S\} \circ \Psi_t^S \, \mathrm{d}t + H_{\geqslant N,\theta} \circ \Psi_1^S. \tag{58}$$

In particular, using the above formulae and $S = S_{<2N,\theta}$, we observe that $\mathcal{C}^{(1)}_{\geqslant 3N,\theta} = 0$ and $\tilde{H}^{(1)}_{\geqslant 3N,\theta} = 0$.

5.6. The parameter domain

In this section we fix the initial set of basic frequencies, $\Lambda \in \bar{\mathcal{U}} = \bar{\mathcal{U}}(R)$ (see (64)), to which we wish to apply the first step of the KAM process. As we want to work iteratively with the analytic dependence with respect to $\Lambda = (\mu, \Omega_2)$, we are forced to complexify μ, Ω_2 and, hence, the corresponding ξ, η. The concrete set of real parameters in which we look for the persistence of 2D tori, of the full system (34), is the set $\mathcal{V} = \mathcal{V}(R)$ in the statement of theorem 3.1 (also see (102)).

To do that, we require some quantitative information on the normal form $\tilde{Z} = \tilde{Z}^{(R)}(x, I, y)$ of theorem 4.1 (see (15)). From (18) we have that $|\tilde{Z}^{(R)}|_{0,R} \leqslant \tilde{c}R^6$, for some $\tilde{c}$ independent of R (eventually it depends on ε and σ but they are kept fixed throughout the paper). By using lemma A.7, we translate this estimate into a bound for the function $Z = Z^{(R)}(u_1, u_2, u_3)$, defined by writing $\tilde{Z}$ in terms of $(q, I, L/2)$, thus obtaining $|Z|_{R^2} \leqslant \tilde{c}R^6$ (see section 2 for the definition of this weighted norm). Then, we conclude that there exists $c_0 > 0$, independent of R, such that

$$|Z|_{R^2} \leqslant c_0 R^6, \quad |\partial_i Z|_{R^2/2} \leqslant c_0 R^4, \quad |\partial^2_{i,j} Z|_{R^2/2} \leqslant c_0 R^2,$$
$$|\partial^3_{i,j,k} Z|_{R^2/2} \leqslant c_0, \qquad i, j, k = 1, 2, 3. \tag{59}$$

To obtain these bounds we use Cauchy estimates over the norm $|\cdot|_{R^2}$. These estimates, together with other properties of the weighted norms used, are shown in appendix A.1. Since all these properties are completely analogous to those of the usual supremum norm, sometimes they are going to be used in the proof without explicit mention.

The first application of bounds (59) is to size up the domain Γ of theorem 4.2.

Lemma 5.1. *With the same hypotheses as theorem 4.2. Let* $0 < c_1 < \min\{1, |d|/(8(1 + a + 2|e|))\}$. *Then, for any* $R > 0$ *small enough, there is a real analytic function* $\mathbb{I} = \mathbb{I}^{(R)}(\zeta)$*, defined on the set*

$$\Gamma = \Gamma(R) := \left\{\zeta = (\xi, \eta) \in \mathbb{C}^2 : |\xi| \leqslant c_1 R^2, \ |\eta| \leqslant c_1 R\right\}, \tag{60}$$

solving equation (22). Moreover, $|\mathbb{I}|_\Gamma \leqslant R^2/4$, $|\partial_\xi \mathbb{I}|_\Gamma \leqslant 2a/|d|$ *and* $|\partial_\eta \mathbb{I}|_\Gamma \leqslant 4c_1 R/|d|$.

Proof. For a fixed $\zeta \in \Gamma$, we consider the function

$$\mathcal{F}(I; \zeta) = \frac{1}{d}(\eta^2 - a\xi - 2e\xi\eta - \partial_1 Z(\xi, I, \xi\eta)),$$

where ζ is dealt as a parameter. We are going to show that, for any R small enough, the function $\mathcal{F}(\cdot;\zeta)$ is a contraction on the set $\{I \in \mathbb{C} : |I| \leqslant R^2/4\}$, uniformly for any $\zeta \in \Gamma$. Then, $\mathbb{I}(\zeta) = \mathcal{F}(\mathbb{I}(\zeta);\zeta)$ is the only fixed point of $\mathcal{F}(\cdot;\zeta)$, with an analytic dependence on ζ. Indeed, using (59) we have

$$|\mathcal{F}(I;\zeta)| \leqslant \frac{1}{|d|}\left(c_1^2R^2 + ac_1R^2 + 2|e|c_1^2R^3 + c_0R^4\right) \leqslant \left(\frac{1+a+2|e|}{|d|}c_1 + \frac{c_0}{|d|}R^2\right)R^2 \leqslant \frac{1}{4}R^2.$$

To ensure the contractive character of $\mathcal{F}(\cdot;\zeta)$ we apply the mean value theorem. Thus, given $\zeta \in \Gamma$ and $|I|, |I'| \leqslant R^2/4$, we have

$$|\mathcal{F}(I';\zeta) - \mathcal{F}(I;\zeta)| = \frac{|\partial_1 Z_1(\xi, I', \xi\eta) - \partial_1 Z_1(\xi, I, \xi\eta)|}{|d|} \leqslant \frac{c_0}{|d|}R^2|I' - I| \leqslant \frac{1}{2}|I' - I|.$$

To finish the proof we only have to control the partial derivatives of $\mathbb{I}$. From the fixed point equation verified by this function, we have

$$\partial_\xi \mathbb{I} = \frac{-a - 2e\eta - \partial^2_{1,1}Z - \eta\partial^2_{1,3}Z}{d + \partial^2_{1,2}Z}, \qquad \partial_\eta \mathbb{I} = \frac{2\eta - 2e\xi - \xi\partial^2_{1,3}Z}{d + \partial^2_{1,2}Z}, \tag{61}$$

with all the partial derivatives of Z evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$. Then the bounds on the derivatives follow straightforwardly. □

Once we have parametrized the family of 2D-bifurcated tori of the normal form as a function of $\zeta = (\xi, \eta)$, we control the corresponding set of basic frequencies $\Lambda = (\mu, \Omega_2)$ as follows.

Lemma 5.2. *With the same hypotheses as lemma 5.1. Let us also assume that* $0 < c_1 < 4a/17$ *and consider* $0 < c_2 < c_1/2$*. Then, for any* $R > 0$ *small enough, there is a real analytic vector-function* $h = h^{(R)}(\Lambda)$*,* $h = (h_1, h_2)$*, defined on the set*

$$\mathcal{U} = \mathcal{U}(R) := \{\Lambda = (\mu, \Omega_2) \in \mathbb{C}^2 : |\mu| \leqslant c_2R,\ |\Omega_2 - \omega_2| \leqslant c_2R\}, \tag{62}$$

solving, with respect to $\zeta = (\xi, \eta)$*, the equations* $\Omega_2 = \Omega_2^{(R)}(\zeta)$ *and* $\mu = \mu^{(R)}(\zeta)$ *defined through (24), (27) and (28), i.e.* $\xi = h_1(\Lambda)$ *and* $\eta = h_2(\Lambda)$*. Moreover,* $|h_1|_\mathcal{U} \leqslant c_1R^2$*,* $|h_2|_\mathcal{U} \leqslant c_1R$*,* $|\partial_\mu h_1|_\mathcal{U} \leqslant 2c_2R/a$*,* $|\partial_\mu h_2|_\mathcal{U} \leqslant 4(|e|/a + |f|/|d|)c_2R$*,* $|\partial_{\Omega_2} h_1|_\mathcal{U} \leqslant 8c_1R/a$ *and* $|\partial_{\Omega_2} h_2|_\mathcal{U} \leqslant 2$*.*

Proof. One proceeds analogously as in the proof of lemma 5.1. By taking into account formulae (24) and (27) we define

$$\mathcal{G}(\zeta;\Lambda) = \left(\frac{\mu^2}{2a} - \frac{2\eta^2}{a} - \frac{\xi}{a}\partial^2_{1,1}Z(\xi, \mathbb{I}(\zeta), \xi\eta), \Omega_2 - \omega_2 - 2c\xi\eta - e\xi \right.$$
$$\left. - f\mathbb{I}(\zeta) - \frac{1}{2}\partial_3 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right).$$

Unfortunately, the function $\mathcal{G}(\cdot;\Lambda)$ is not contractive in general. Then, we introduce

$$\mathcal{F}(\zeta;\Lambda) = (\mathcal{G}_1(\zeta;\Lambda), \mathcal{G}_2(\mathcal{G}_1(\zeta;\Lambda), \eta;\Lambda)).$$

We are going to verify that the function $\mathcal{F}(\cdot;\Lambda)$ is a contraction on $\Gamma = \Gamma(R)$ (see (60)), uniformly on $\Lambda \in \mathcal{U}$. Then, $h(\Lambda) = \mathcal{F}(h(\Lambda);\Lambda)$ is the only fixed point of $\mathcal{F}(\cdot;\Lambda)$. To do that, we make R as small as necessary and use the bounds on Z, $\mathbb{I}$ and their partial derivatives given by (59) and lemma 5.1. First, we have

$$|\mathcal{F}_1| \leqslant \frac{1}{a}\left(\frac{c_2^2}{2} + 2c_1^2 + c_0c_1R^2\right)R^2 \leqslant c_1R^2.$$

After that we apply this bound to $\mathcal{F}_2$, thus obtaining

$$|\mathcal{F}_2| \leqslant \left(c_2 + 2|c|c_1^2 R^2 + |e|c_1 R + \frac{|f|}{4} R + \frac{c_0}{2} R^3\right) R \leqslant c_1 R.$$

Now we check that $\mathcal{F}(\cdot; \Lambda)$ is a contraction on Γ. For $\mathcal{F}_1$ we have

$$\begin{aligned}
|\mathcal{F}_1(\zeta'; \Lambda) - \mathcal{F}_1(\zeta; \Lambda)| &\leqslant \frac{2}{a}|(\eta')^2 - \eta^2| + \frac{|\xi' - \xi|}{a}|\partial_{1,1}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)| \\
&\quad + \frac{|\xi|}{a}|\partial_{1,1}^2 Z(\xi', \mathbb{I}(\zeta'), \xi'\eta') - \partial_{1,1}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)| \\
&\leqslant \frac{4c_1}{a} R|\eta' - \eta| + \frac{c_0}{a}(1 + c_1)R^2|\xi' - \xi| \\
&\quad + \frac{c_0 c_1}{a} R^2(|\mathbb{I}(\zeta') - \mathbb{I}(\zeta)| + |\xi'\eta' - \xi\eta|) \\
&\leqslant \left(\frac{4c_1}{a} R + \frac{c_0 c_1}{a} R^3 \left(\frac{4c_1}{|d|} + c_1 R\right)\right)|\eta' - \eta| \\
&\quad + \left(\frac{c_0}{a} R^2 + \frac{c_0 c_1}{a} R^2 \left(1 + \frac{2a}{|d|} + c_1 R\right)\right)|\xi' - \xi| \\
&\leqslant \min\left\{\frac{1}{2}, \frac{|d|}{8(|e||d| + 2a|f|)}\right\}|\zeta' - \zeta|.
\end{aligned} \tag{63}$$

Similarly, we obtain the next bound for $\mathcal{G}_2$:

$$\begin{aligned}
|\mathcal{G}_2(\zeta'; \Lambda) - \mathcal{G}_2(\zeta; \Lambda)| &\leqslant \left(2|c|c_1 R^2 + \frac{4|f|c_1}{|d|} R + \frac{c_0}{2} R^3 \left(\frac{4c_1}{|d|} + c_1 R\right)\right)|\eta' - \eta| \\
&\quad + \left(2|c|c_1 R + |e| + \frac{2a|f|}{|d|} + \frac{c_0}{2} R^2 \left(1 + \frac{4a}{|d|} + c_1 R\right)\right)|\xi' - \xi|.
\end{aligned}$$

Hence, going back to the definition of $\mathcal{F}_2$ and using inequality (63), we have

$$|\mathcal{F}_2(\zeta'; \Lambda) - \mathcal{F}_2(\zeta; \Lambda)| \leqslant \tfrac{1}{2}|\zeta' - \zeta|.$$

Finally, the bounds on the partial derivatives of $h(\Lambda)$ follow at once by computing the derivatives of the fixed point equation $h(\Lambda) = \mathcal{G}(h(\Lambda); \Lambda)$ (compare (61)). □

At this point we can express the remaining intrinsic frequency Ω_1 in terms of Λ. The next lemma accounts for this dependence.

Lemma 5.3. *With the same hypotheses as lemma 5.2. Let us define* $\Omega_1^{(0)} = \Omega_1^{(0,R)}(\Lambda)$ *as the function* $\Omega_1 = \Omega_1^{(R)}(\zeta)$ *of (23), when expressed in terms of* Λ *through the change* $\zeta = h^{(R)}(\Lambda)$ *given by lemma 5.2 (see (30)). There exists a constant* $c_3 > 0$*, independent of* R*, such that if* $R > 0$ *is small enough then*

$$|\Omega_1^{(0)} - \omega_1|_{\mathcal{U}} \leqslant c_3 R^2, \quad |\partial_\mu \Omega_1^{(0)}|_{\mathcal{U}} \leqslant c_3 R, \quad |\partial_{\Omega_2} \Omega_1^{(0)}|_{\mathcal{U}} \leqslant c_3 R, \quad \mathrm{Lip}_{\mathcal{U}}(\Omega_1^{(0)}) \leqslant 2c_3 R,$$

where the set $\mathcal{U} = \mathcal{U}^{(R)}$ *is defined in (62).*

Proof. These bounds are straightforward from those of lemmas 5.1 and 5.2. We leave the details for the reader. In particular, to derive the Lipschitz estimate we note that $\mathcal{U}$ is a convex set. □

Now we introduce the initial set of (complex) parameters $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{U}} = \bar{\mathcal{U}}(R)$ we are going to consider for the KAM scheme of sections 5.3 and 5.5. We define

$$\begin{aligned}
\bar{\mathcal{U}} := \big\{\Lambda \in \mathbb{C}^2 : (M^{(0)})^{\alpha/2} \leqslant \mathrm{Re}\,\mu,\ |\mathrm{Im}\,\mu| \leqslant 2(M^{(0)})^{\alpha/2},\ |\mu| \leqslant c_2 R, \\
|\Omega_2 - \omega_2| \leqslant c_2 R,\ |\mathrm{Im}\,\Omega_2| \leqslant 2(M^{(0)})^{\alpha/2},\ |h_1(\Lambda)| \geqslant (M^{(0)})^{\alpha/2}\big\},
\end{aligned} \tag{64}$$

where α, $M^{(0)} = M^{(0)}(R)$ are thus in the statement of theorem 3.1 and c_2, $h = h^{(R)} = (h_1^{(R)}, h_2^{(R)})$ are given by lemma 5.2.

Remark 5.3. The set $\bar{\mathcal{U}}$ is the restriction to $\mathbb{R}^+ \times \mathbb{R}$ of the set $\mathcal{U}$ provided by lemma 5.2—i.e. the set $\mathcal{V}$ of (102)—plus a small complex widening and small technical restrictions. We point out that $\bar{\mathcal{U}} \subset \mathcal{U}$, for small R. In particular, definition (64) implies that if $\Lambda \in \bar{\mathcal{U}}$, then $|\mu| \geqslant (M^{(0)})^{\alpha/2}$—i.e. we are 'far' from parabolic tori given by $\mu = 0$—and $\zeta = (\xi, \eta) = h(\Lambda)$ verifies $(M^{(0)})^{\alpha/2} \leqslant |\xi| \leqslant c_1 R^2$—i.e. we are 'far' from the stable periodic orbits of the family given by $\xi = 0$—and $|\eta| \leqslant c_1 R$ (see lemma 5.1). See figure 1.

5.7. Domain of definition of the Λ family of Hamiltonians

After that, we consider the 'initial' family of Hamiltonian systems $H_\Lambda^{(0)}$ (see (34)) and fix their domain of definition.

Taking into account the symplectic changes (19) and (32), one may write the normal form coordinates $(\theta, x_1, x_2, I, y_1, y_2)$ of (14) as a function of $(\theta_1, \theta_2, x, I_1, I_2, y)$, depending on the prefixed R and the parameter Λ. Writing them up explicitly, we have

$$\theta = \theta_1 + \frac{\xi}{\lambda_+}(d + \partial_{1,2}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)) \left(\frac{x}{\xi} + \frac{y}{\lambda_+}\right), \qquad I = \mathbb{I}(\zeta) + I_1,$$

$$y_1 = \sqrt{2\xi(1+\hat{q})}\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right)\cos\phi_2 - \frac{2\xi\eta + I_2}{\sqrt{2\xi(1+\hat{q})}}\sin\phi_2, \qquad x_1 = \sqrt{2\xi(1+\hat{q})}\cos\phi_2,$$

$$y_2 = -\sqrt{2\xi(1+\hat{q})}\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right)\sin\phi_2 - \frac{2\xi\eta + I_2}{\sqrt{2\xi(1+\hat{q})}}\cos\phi_2, \qquad x_2 = -\sqrt{2\xi(1+\hat{q})}\sin\phi_2,$$

with

$$\phi_2 = \theta_2 + \frac{1}{\lambda_+}\left(-\eta + e\xi + \frac{\xi}{2}\partial_{1,3}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right)\left(\frac{x}{\xi} + \frac{y}{\lambda_+}\right),$$

$$\hat{q} = \frac{x}{\xi} - \frac{y}{\lambda_+} + \frac{2}{\lambda_+^2}(d + \partial_{1,2}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta))I_1 + \frac{2}{\xi\lambda_+^2}\left(-\eta + e\xi + \frac{\xi}{2}\partial_{1,3}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right)I_2,$$

where $\lambda_+ = \mathrm{i}\mu$ and $\zeta = (\xi, \eta)$ are functions of Λ through lemma 5.2. See also theorem 4.2 and lemma 5.1 for the definition and bounds on $\mathbb{I}$.

In view of this coordinate transformation, we select (see theorem 4.1)

$$\rho^{(0)} := \min\{\sigma^{-2}\rho_0/4, \log(2)/2\}, \qquad R^{(0)} = R^{(0)}(R) := (M^{(0)}(R))^\alpha, \tag{65}$$

and we want to show that the change $(\theta_1, \theta_2, x, I_1, I_2, y) \mapsto (\theta, x_1, x_2, I, y_1, y_2)$ is well defined from $\mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$ to $\mathcal{D}_{1,2}(\sigma^{-2}\rho_0/2, R)$ (see (1)), for any $\Lambda \in \bar{\mathcal{U}}$, and controlled in terms of the weighted norm $|\cdot|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}}$—computed by expanding it in $(\theta_1, \theta_2, x, I_1, I_2, y)$.

To do that, first of all we combine the definition of $\bar{\mathcal{U}}$ in (64) (see also remark 5.3) with the bounds (59) on the partial derivatives of $Z = Z^{(R)}$ to obtain, for any R small enough, the following estimates (we recall that $M^{(0)}(R)$ goes to zero faster than any power of R):

$$\left|\frac{x}{\xi} + \frac{y}{\lambda_+}\right|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant 2(M^{(0)})^{\alpha/2}, \qquad \left|\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant c_2 R\frac{(M^{(0)})^{3\alpha/4}}{\sqrt{|\xi|}},$$

$$|\theta - \theta_1|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant 2c_1 R^2(|d| + c_0 R^2) < \sigma^{-2}\rho_0/4,$$

$$|\phi_2 - \theta_2|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant 2c_1 R + 2|e|c_1 R^2 + c_0 c_1 R^4 < \log(2)/2,$$

$$|\hat{q}|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant 2(|d| + c_0 R^2)(M^{(0)})^\alpha + (2 + 2c_1 R + 2|e|c_1 R^2 + c_0 c_1 R^4)(M^{(0)})^{\alpha/2} \leqslant 3(M^{(0)})^{\alpha/2}.$$

By assuming $3(M^{(0)})^{\alpha/2} \leqslant 1/2$, we use the estimate on $\hat{q}$ and lemma A.6 to obtain

$$|\sqrt{1+\hat{q}}|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant (4 - \sqrt{2})/2, \qquad |(\sqrt{1+\hat{q}})^{-1}|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant \sqrt{2}.$$

The estimate on ϕ_2, combined with lemma A.2, gives

$$|\sin\phi_2|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}} \leqslant \mathrm{e}^{\rho^{(0)}+|\phi_2-\theta_2|_{\bar{\mathcal{U}},0,R^{(0)}}} \leqslant \mathrm{e}^{\log(2)} = 2,$$

and the same holds for $\cos\phi_2$. Moreover we also have, for small R, $|2\xi\eta + I_2|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant 3c_1\sqrt{c_1}R^2\sqrt{|\xi|}$.

If we put these bounds all together we obtain, for $j = 1, 2$,

$$|I - I_1|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant \frac{R^2}{4}, \qquad |x_j|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}} \leqslant 2(2\sqrt{2}-1)\sqrt{c_1}R, \qquad |y_j|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}} \leqslant 7c_1\sqrt{c_1}R^2.$$

Consequently, if R is small enough and c_1 is such that $2(2\sqrt{2}-1)\sqrt{c_1} < 1$, then we can ensure that the transformation is controlled as we claimed.

Now, we are ready to bound the different elements of the initial system $H^{(0)} = H_\Lambda^{(0)}$ in (34) in the domain $\mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$. To do that we introduce a constant $\kappa > 0$, independent of R, defined so that we achieve conditions below. Moreover, we take strong advantage of the use of weighted norms in order to control any term of the decomposition (34) by the norm of the full system. We have (see theorem 4.1)

$$|H^{(0)}|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}} \leqslant \kappa, \qquad |\tilde{H}^{(0)}|_{\bar{\mathcal{U}},0,R^{(0)}} \leqslant \kappa, \qquad |\hat{H}^{(0)}|_{\bar{\mathcal{U}},\rho^{(0)},R^{(0)}} \leqslant M^{(0)}. \tag{66}$$

By using formulae (36) we also have

$$|\mathcal{C}_{1,1}^{(0)}|_{\bar{\mathcal{U}}} \leqslant \frac{\kappa}{(M^{(0)})^{\alpha}}, \quad |\mathcal{C}_{1,2}^{(0)}|_{\bar{\mathcal{U}}} \leqslant \frac{\kappa}{(M^{(0)})^{\alpha}}, \quad |\mathcal{C}_{2,2}^{(0)}|_{\bar{\mathcal{U}}} \leqslant \frac{\kappa}{(M^{(0)})^{3\alpha/2}}, \quad |\mathcal{C}^{(0)}|_{\bar{\mathcal{U}}} \leqslant \frac{\kappa}{(M^{(0)})^{3\alpha/2}}. \tag{67}$$

Finally, we consider the matrix $\bar{\mathcal{A}}^{(0)}$ discussed in section 5.4, whose determinant defines the non-degenerate character of the selected set of basic frequencies. For this matrix we have proved that $|\det\bar{\mathcal{A}}^{(0)}|_{\mathcal{U}} \geqslant |ad|/(2|\mu|^3)$, for any R small enough. Then, using again the closed formulae (36) for $\mathcal{C}_{1,2}^{(0)}$, $\mathcal{C}_{2,2}^{(0)}$ and those on the partial derivatives on $\tilde{H}^{(0)}$ given in section 5.4, we have

$$|(\bar{\mathcal{A}}^{(0)})^{-1}|_{\bar{\mathcal{U}}} \leqslant \frac{\kappa}{(M^{(0)})^{\alpha/2}}. \tag{68}$$

5.8. The iterative lemma

The purpose of this section is to give quantitative estimates on the effect of one step of the iterative process described in sections 5.3 and 5.5. The result controlling this process is stated as follows.

Lemma 5.4 (Iterative lemma). *We consider a family of Hamiltonian systems $H = H_\Lambda(\theta, x, I, y)$ defined in $\mathcal{D}_{2,1}(\bar{\rho}, \bar{R})$ for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}} \subset \mathbb{C}^2$, for some $\bar{\rho}, \bar{R} \in (0, 1)$, with an analytic dependence on all variables and parameters. The Hamiltonian H takes the form (with everything depending on Λ)*

$$H = \phi + \langle\Omega, I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}z\rangle + \tfrac{1}{2}\langle I, \mathcal{C}(\theta)I\rangle + \tilde{H}(\theta, x, I, y) + \hat{H}(\theta, x, I, y),$$

where $\mathcal{B}$ is defined in (35), being $\lambda_+ = \mathrm{i}\mu$, the function $\tilde{H}$ contains 'higher order terms', i.e. $\tilde{H} = [\tilde{H}]$ (see (4)) and (abusing notation) $\Omega(\Lambda) = (\Omega_1(\Lambda), \Omega_2)$. We suppose that there is an integer $\bar{N} \geqslant 1$ and real quantities $\tau > 1$, $0 < \alpha < 1$, $\kappa > 0$, $0 < \bar{M} \leqslant M^{(0)} < 1$ so that $(M^{(0)})^\alpha/2 \leqslant \bar{R} \leqslant (M^{(0)})^\alpha$ and, for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}}$, we have $(H - \hat{H})_{\geqslant\bar{N},\theta} = 0$, $|\mu| \geqslant (M^{(0)})^{\alpha/2}$,

$$|\langle k, \Omega\rangle + \ell\mu| \geqslant (M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \qquad k \in \mathbb{Z}^2, \quad 0 < |k|_1 < 2\bar{N}, \quad \ell \in \{0, 1, 2\}, \tag{69}$$

and the following bounds:

$$|H|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant \kappa, \qquad |\tilde{H}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant 2\kappa, \qquad |\hat{H}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant \bar{M},$$
$$|\mathcal{C}|_{\bar{\mathcal{E}},\bar{\rho},0} \leqslant \frac{2\kappa}{(M^{(0)})^{3\alpha/2}}, \qquad |(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leqslant \frac{2\kappa}{(M^{(0)})^{\alpha/2}},$$

where $\bar{\mathcal{A}}$ denotes the matrix $\mathcal{A}$ defined in (45) by setting $\Xi = \tilde{H}$ and $C = \mathcal{C}$.

Under these conditions, given $0 < \rho^{(\infty)} < \bar{\rho}$, there is a constant $\bar{\kappa} \geqslant 1$, depending only on κ, τ and $\rho^{(\infty)}$, such that if for certain $0 < \bar{\delta} < 1/2$ we have $\bar{\rho}^{(1)} := \bar{\rho} - 6\bar{\delta} \geqslant \rho^{(\infty)}$ and

$$\frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{14\alpha}} \leqslant 1, \tag{70}$$

then for any $\Lambda \in \bar{\mathcal{E}}$ there exists a canonical transformation $\Psi = \Psi_\Lambda(\theta, x, I, y)$, with an analytic dependence on all variables and parameters, acting as $\Psi : \mathcal{D}_{2,1}(\bar{\rho}^{(1)}, \bar{R}^{(1)}) \to \mathcal{D}_{2,1}(\bar{\rho} - 5\bar{\delta}, \bar{R}\exp(-2\bar{\delta}))$, with $\bar{R}^{(1)} := \bar{R}\exp(-3\bar{\delta})$. If $\Psi - \mathrm{Id} = (\Theta, \mathcal{X}, \mathcal{I}, \mathcal{Y})$, then all the components are 2π-periodic in θ and verify

$$|\Theta|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}} \leqslant \bar{\delta}, \tag{71}$$

$$|\mathcal{I}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+2}(M^{(0)})^{15\alpha/2}} \leqslant (\bar{R}\exp(-2\bar{\delta}))^2 - (\bar{R}\exp(-3\bar{\delta}))^2, \tag{72}$$

$$|\mathcal{Z}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{17\alpha/2}} \leqslant \bar{R}\exp(-2\bar{\delta}) - \bar{R}\exp(-3\bar{\delta}), \tag{73}$$

with $\mathcal{Z} = (\mathcal{X}, \mathcal{Y})$. This canonical transformation is defined so that we can expand the transformed Hamiltonian H by the action of Ψ, $H^{(1)} = H^{(1)}_\Lambda(\theta, x, I, y)$, as

$$H^{(1)} := H \circ \Psi = \phi^{(1)} + \langle \Omega^{(1)}, I\rangle + \tfrac{1}{2}\langle z, \mathcal{B}z\rangle + \tfrac{1}{2}\langle I, \mathcal{C}^{(1)}(\theta) I\rangle$$
$$+\tilde{H}^{(1)}(\theta, x, I, y) + \hat{H}^{(1)}(\theta, x, I, y),$$

with everything depending on Λ, where $\Omega_2^{(1)}(\Lambda) = \Omega_2$, $[\tilde{H}^{(1)}] = \tilde{H}^{(1)}$, $(H^{(1)} - \hat{H}^{(1)})_{\geqslant 3\bar{N},\theta} = 0$ and

$$|\Omega_1^{(1)} - \Omega_1|_{\bar{\mathcal{E}}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau}(M^{(0)})^{13\alpha/2}}, \qquad |\tilde{H}^{(1)} - \tilde{H}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}},$$

$$|\mathcal{C}^{(1)} - \mathcal{C}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},0} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{27\alpha/2}}, \qquad |(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{29\alpha/2}},$$

$$|H^{(1)}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \kappa, \qquad |\hat{H}^{(1)}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}} \leqslant \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{4\tau+6}(M^{(0)})^{19\alpha}} + \bar{M}\exp(-\bar{\delta}\bar{N}),$$

where $\bar{\mathcal{A}}^{(1)}$ is defined analogously to $\bar{\mathcal{A}}$.

Remark 5.4. It is not difficult to realize that if the (complex) analytic Hamiltonian H of the statement verifies the symmetries due to the complexification (33), then the same holds for $H^{(1)}$ (see section 5.2 and remark 5.2 for more details).

Proof. Our plan is to give only a sketch of the proof. The full details can be easily developed by hand by the interested reader. During the proof, and abusing notation, the constant $\bar{\kappa}$ will be redefined several times in order to meet a finite number of conditions. The constant $\bar{\kappa}$ of the statement is the final one. Moreover, we will use some technical lemmas given in appendix A.1 in order to control the weighted norms of the derivatives (Cauchy estimates), composition of functions and solutions of small divisor equations, without an explicit mention. Finally, the analytic dependence of Ψ_Λ on Λ, and so of $H^{(1)}_\Lambda$, follows straightforwardly from the way in which this canonical transformation is generated.

We start by decomposing $H = H_{<\bar N,\theta} + H_{\geqslant \bar N,\theta}$ as in (49), expanding $H_{<\bar N,\theta}$ as in (38) and defining $\bar H_{<\bar N,\theta}$ from $H_{<\bar N,\theta}$ as in (39). After that, we compute the generating function $S = S_\Lambda(\theta, x, I, y)$ of (40), defined by solving the homological equations (eq1)–(eq5). Then we define the canonical transformation $\Psi = \Psi_\Lambda$ as the time one flow of the Hamiltonian system S, i.e. $\Psi = \Psi^S_{t=1}$. We recall that the condition $(H^{(1)} - \hat H^{(1)})_{\geqslant 3\bar N,\theta} = 0$ follows at once using (57), (58) and $S = S_{<2N,\theta}$.

After that, we perform the quantitative part of the lemma. First of all, we have the following bounds for the terms of decomposition (38) of $H_{<\bar N,\theta}$ and for $H_{\geqslant \bar N,\theta} = \hat H_{\geqslant \bar N,\theta}$,

$$|\mathrm{a} - \phi|_{\bar{\mathcal E},\bar\rho,0} \leqslant \bar M, \qquad |\mathrm{b}|_{\bar{\mathcal E},\bar\rho,0} \leqslant \frac{\bar M}{\bar R}, \qquad |\mathrm{c} - \Omega|_{\bar{\mathcal E},\bar\rho,0} \leqslant \frac{\bar M}{\bar R^2},$$

$$|\mathrm{B} - \mathcal B|_{\bar{\mathcal E},\bar\rho,0} \leqslant 8\frac{\bar M}{\bar R^2}, \qquad |\mathrm{C} - \mathcal C|_{\bar{\mathcal E},\bar\rho,0} \leqslant 8\frac{\bar M}{\bar R^4}, \qquad |\mathrm{E}|_{\bar{\mathcal E},\bar\rho,0} \leqslant 2\frac{\bar M}{\bar R^3},$$

$$|\Xi - \tilde H|_{\bar{\mathcal E},\bar\rho,\bar R} \leqslant \bar M, \qquad |H_{\geqslant \bar N,\theta}|_{\bar{\mathcal E},\bar\rho-\bar\delta,\bar R} \leqslant \bar M \exp(-\bar\delta \bar N).$$

By assuming $\bar\kappa \bar M/(M^{(0)})^{5\alpha/2} \leqslant 1$ we also have

$$|\mathrm{C}|_{\bar{\mathcal E},\bar\rho,0} \leqslant \frac{4\kappa}{(M^{(0)})^{3\alpha/2}}, \qquad |\Xi|_{\bar{\mathcal E},\bar\rho,\bar R} \leqslant 4\kappa.$$

Furthermore, we need to control the norm of $(\langle \mathcal A\rangle_\theta)^{-1}$, where $\mathcal A$ is defined in (45). We observe that

$$(\langle \mathcal A\rangle_\theta)^{-1} - (\langle \bar{\mathcal A}\rangle_\theta)^{-1} = -\left(\mathrm{Id} + (\langle \bar{\mathcal A}\rangle_\theta)^{-1}(\langle \mathcal A\rangle_\theta - \langle \bar{\mathcal A}\rangle_\theta)\right)^{-1} (\langle \bar{\mathcal A}\rangle_\theta)^{-1}(\langle \mathcal A\rangle_\theta - \langle \bar{\mathcal A}\rangle_\theta)(\langle \bar{\mathcal A}\rangle_\theta)^{-1}. \tag{74}$$

If we also assume $\bar\kappa \bar M/(M^{(0)})^{9\alpha/2} \leqslant 1$, then we have

$$|\mathcal A - \bar{\mathcal A}|_{\bar{\mathcal E},\bar\rho,0} \leqslant 16\frac{\bar M}{\bar R^4}, \qquad |(\langle \bar{\mathcal A}\rangle_\theta)^{-1}|_{\bar{\mathcal E}}\, |\langle \mathcal A\rangle_\theta - \langle \bar{\mathcal A}\rangle_\theta|_{\bar{\mathcal E}} \leqslant \frac{1}{2},$$

$$|(\langle \mathcal A\rangle_\theta)^{-1} - (\langle \bar{\mathcal A}\rangle_\theta)^{-1}|_{\bar{\mathcal E}} \leqslant \frac{\bar\kappa \bar M}{(M^{(0)})^{5\alpha}}, \qquad |(\langle \mathcal A\rangle_\theta)^{-1}|_{\bar{\mathcal E}} \leqslant \frac{4\kappa}{(M^{(0)})^{\alpha/2}}. \tag{75}$$

Remark 5.5. To estimate the difference $\mathcal A - \bar{\mathcal A}$, we take into account that the partial derivatives of $\Xi - \tilde H$ are evaluated at $z = 0$ and $I = 0$ when bounding them by means of Cauchy estimates.

Now we bound the solutions of the homological equations, which are displayed explicitly from (50) to (55) (see also the compatibility equation (44)). For instance, we have the following bound for d (see (69) and lemma A.4):

$$|\mathrm{d}|_{\bar{\mathcal E},\bar\rho-\bar\delta,0} \leqslant \left(\frac{\tau}{\bar\delta \exp(1)}\right)^\tau \frac{|\{a\}_\theta|_{\bar{\mathcal E},\bar\rho,0}}{(M^{(0)})^{\alpha/2}} \leqslant \left(\frac{\tau}{\bar\delta \exp(1)}\right)^\tau \frac{|a - \phi|_{\bar{\mathcal E},\bar\rho,0}}{(M^{(0)})^{\alpha/2}} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{\alpha/2}}.$$

Similarly, we bound (recursively) the remaining ingredients involved in the resolution of these equations (see (41)–(43), (46), (47) and (56)), thus obtaining

$$|\mathrm{e}|_{\bar{\mathcal E},\bar\rho-\bar\delta,0} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{3\alpha/2}}, \qquad |\mathrm{h}_1|_{\bar{\mathcal E}} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{2\alpha}}, \qquad |\mathrm{h}_2|_{\bar{\mathcal E}} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{9\alpha/2}},$$

$$|\chi|_{\bar{\mathcal E}} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{5\alpha}}, \qquad |\Omega_1^{(1)} - \Omega_1|_{\bar{\mathcal E}} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^\tau (M^{(0)})^{13\alpha/2}}, \qquad |\tilde{\mathrm{c}}|_{\bar{\mathcal E},\bar\rho-2\bar\delta,0} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^{\tau+1} (M^{(0)})^{13\alpha/2}},$$

$$|\mathrm{f}|_{\bar{\mathcal E},\bar\rho-3\bar\delta,0} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^{2\tau+1} (M^{(0)})^{7\alpha}}, \qquad |\tilde{\mathrm{B}} - \mathcal B|_{\bar{\mathcal E},\bar\rho-2\bar\delta,0} \leqslant \frac{\bar\kappa \bar M}{\bar\delta^{\tau+1} (M^{(0)})^{9\alpha}},$$

$$|\langle \mathrm{G}\rangle_\theta|_{\bar{\mathcal{E}}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{19\alpha/2}}, \qquad |\mathrm{G}|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},0} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}},$$

$$|\tilde{\mathrm{E}}|_{\bar{\mathcal{E}},\bar{\rho}-2\bar{\delta},0} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{10\alpha}}, \qquad |\mathrm{F}|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},0} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{21\alpha/2}}.$$

Remark 5.6. Besides the indication pointed out in remark 5.5, we have also used a similar idea to bound the average $\langle\cdot\rangle_\theta$ of any expression containing derivatives with respect to θ, i.e. $|\langle\partial_\theta(\cdot)\rangle_\theta| \leqslant |\cdot|_{\rho,0}/(\rho\exp(1))$.

From here we have (see (40))

$$|\nabla_\theta S|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+2}(M^{(0)})^{15\alpha/2}}, \qquad |\nabla_I S|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},\bar{R}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}},$$

$$|\nabla_z S|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},\bar{R}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{17\alpha/2}}.$$

Now we apply lemma A.3 to obtain the leftmost parts of estimates (71), (72) and (73) on the components of the canonical change $\Psi = \Psi_1^S$. Then, the rightmost parts of these estimates follow at once. For instance, for (72) we have

$$\frac{|\mathcal{I}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}}}{(\bar{R}\exp(-2\bar{\delta}))^2-(\bar{R}\exp(-3\bar{\delta}))^2} = \frac{\exp(4\bar{\delta})|\mathcal{I}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}}}{\bar{R}^2(1-\exp(-2\bar{\delta}))} \leqslant \frac{4\exp(2)\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}} \leqslant 1, \tag{76}$$

which is guaranteed by (70). Here, we have used that $\bar{R} \geqslant (M^{(0)})^\alpha/2$, $0 < \bar{\delta} \leqslant 1/2$ and the bound $(1-\exp(-x))^{-1} \leqslant 2/x$, whenever $0 < x \leqslant 1$. Similarly we obtain the rightmost parts of (71) and (73).

To finish the proof it only remains to bound the transformed system $H^{(1)}$. Concretely, we have to focus on formulae (57) and (58). First we observe that

$$|\bar{H}_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant |H_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} = |H-\hat{H}_{\geqslant\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant \kappa+\bar{M} \leqslant 2\kappa,$$
$$|H_{<\bar{N},\theta}-\bar{H}_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant |\hat{H}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leqslant \bar{M}$$

and hence

$$|\{\bar{H}_{<\bar{N},\theta},S\}|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}\exp(-\bar{\delta})} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}},$$

$$|\{\{\bar{H}_{<\bar{N},\theta},S\},S\}|_{\bar{\mathcal{E}},\bar{\rho}-5\bar{\delta},\bar{R}\exp(-2\bar{\delta})} \leqslant \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{4\tau+6}(M^{(0)})^{19\alpha}},$$

$$|\{H_{<\bar{N},\theta}-\bar{H}_{<\bar{N},\theta},S\}|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}\exp(-\bar{\delta})} \leqslant \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}}.$$

From these bounds we easily derive the estimates for $H^{(1)}$ in the statement, for a suitable $\bar{\kappa}$. The only one that is not immediate is thus on $(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta)^{-1}-(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}$. To obtain this bound we proceed as in (74) and (75). Indeed,

$$|\bar{\mathcal{A}}^{(1)}-\bar{\mathcal{A}}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},0} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{27\alpha/2}}, \qquad |(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}}\,|\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta-\langle\bar{\mathcal{A}}\rangle_\theta|_{\bar{\mathcal{E}}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{14\alpha}},$$

$$\left|\left(\mathrm{Id}+(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta-\langle\bar{\mathcal{A}}\rangle_\theta)\right)^{-1}\right|_{\bar{\mathcal{E}}} \leqslant 2, \quad |(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta)^{-1}-(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leqslant \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{29\alpha/2}}.$$

The control of the above expressions induces the strongest restriction when defining condition (70). □

5.9. Convergence of the iterative scheme

Now we have all the ingredients needed to prove the convergence of the iterative (KAM) scheme of sections 5.3 and 5.5. Concretely, we consider the sequence of transformed Hamiltonians $H^{(n)} = H_\Lambda^{(n)}$—starting with $H_\Lambda^{(0)}$ of (34)—and we want this sequence to converge to the 'normalized' Hamiltonian $H^{(\infty)} = H_\Lambda^{(\infty)}$ of (37) if $\Lambda = (\mu, \Omega_2)$ belongs to a suitable (Cantor) set $\bar{\mathcal{E}}^{(\infty)}$ (see (82)). This limit Hamiltonian has, for any $\Lambda \in \bar{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2$, an invariant 2D torus with a vector of basic frequencies Λ.

To construct this sequence we iteratively apply lemma 5.4, so that we define $H^{(n+1)} = H^{(n)} \circ \Psi^{(n)}$, where $\Psi^{(n)} = \Psi_\Lambda^{(n)}$ is the canonical transformation provided by the lemma. Of course, all of this process depends on the value of R we have fixed at the beginning of section 5 and, at any step, everything is analytic on Λ, in a (complex) set $\bar{\mathcal{E}}^{(n)}$ shrinking with n (see (81)). Therefore, to ensure the inductive applicability of lemma 5.4 we have to control, at every step, the conditions of the statement.

First of all we observe that the constants τ, α and the function $M^{(0)} = M^{(0)}(R)$ (see (17)) have been clearly set during the paper, whilst the constant κ (independent of R) is the one introduced at the end of section 5.7.

Now we select a fixed $0 < \bar{\delta}^{(0)} < 1/2$ (independent of R) and introduce (see (65))

$$\rho^{(\infty)} := \rho^{(0)} - 13\bar{\delta}^{(0)}, \qquad R^{(\infty)} = R^{(\infty)}(R) := R^{(0)}(R)\exp(-7\bar{\delta}^{(0)}). \tag{77}$$

We also assume $\bar{\delta}^{(0)}$ small enough so that $\rho^{(\infty)} > 0$ and $\exp(7\bar{\delta}^{(0)}) \leqslant 2$. Hence, $R^{(\infty)} \geqslant (M^{(0)})^\alpha/2$. We use $\bar{\delta}^{(0)}$ to define, recursively,

$$\bar{\delta}^{(n)} := \bar{\delta}^{(0)}/2^n, \qquad \bar{\rho}^{(n+1)} := \bar{\rho}^{(n)} - 6\bar{\delta}^{(n)}, \qquad \bar{R}^{(n+1)} := \bar{R}^{(n)}\exp(-3\bar{\delta}^{(n)}), \quad n \geqslant 0, \tag{78}$$

starting with $\bar{\rho}^{(0)} := \rho^{(0)}$ and $\bar{R}^{(0)} := R^{(0)}$. Hence, $\bar{R}^{(n)}$ depends on the prefixed R. Our purpose is to apply lemma 5.4 to $H^{(n)}$ with $\bar{\rho} = \bar{\rho}^{(n)}$ and $\bar{\delta} = \bar{\delta}^{(n)}$.

After that we set the value of $\bar{N}$ at the nth step of the iterative process. To do that, we consider the bound for the size of the 'error term' $\hat{H}^{(1)}$ of the transformed Hamiltonian provided by the iterative lemma. Then, this expression suggests selecting $\bar{N} = \bar{N}^{(n)}(R) \in \mathbb{N}$ so that

$$\bar{M}^{(n)}\exp(-\bar{\delta}^{(n)}\bar{N}^{(n)}) \leqslant \frac{\bar{\kappa}(\bar{M}^{(n)})^2}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}. \tag{79}$$

This implies that we can define after this nth stage

$$\bar{M}^{(n+1)} = \frac{2\bar{\kappa}(\bar{M}^{(n)})^2}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}, \tag{80}$$

so that $|\hat{H}^{(n+1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} \leqslant \bar{M}^{(n+1)}$, starting with $\bar{M}^{(0)} := M^{(0)}$.

Moreover, we also note that to define $\bar{N}^{(n)}$ we have to take care of the inductive condition $(H^{(n)} - \hat{H}^{(n)})_{\geqslant \bar{N}^{(n)},\theta} = 0$. Assuming that it is true at the nth step and using that the transformed Hamiltonian verifies $(H^{(n+1)} - \hat{H}^{(n+1)})_{\geqslant 3\bar{N}^{(n)},\theta} = 0$, then, to keep track of it, we only need to ensure that $\bar{N}^{(n+1)} > 3\bar{N}^{(n)}$ (see (85) and comments below).

Finally, we introduce the set $\bar{\mathcal{E}} = \bar{\mathcal{E}}^{(n)}(R)$ we dealt with at any step. This set is defined recursively from $\bar{\mathcal{U}}(R)$ (see (64)), by taking into account the Diophantine conditions (69). Concretely, we first introduce, for convenience, $\bar{\mathcal{E}}^{(-1)} = \bar{\mathcal{U}}$ and, for each $n \geqslant 0$,

$$\begin{aligned}\bar{\mathcal{E}}^{(n)} := \{\Lambda \in \bar{\mathcal{E}}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geqslant a_n (M^{(0)})^{\alpha/2}|k|_1^{-\tau},\\ 0 < |k|_1 < 2\bar{N}^{(n)},\ \ell \in \{0,1,2\}\},\end{aligned} \tag{81}$$

with $a_n = 1 + 2^{-n}$, where $\Omega_1^{(n)}(\Lambda)$ is defined recursively (starting with $\Omega_1^{(0)}(\Lambda)$ given by lemma 5.3) and $\Omega_2^{(n)}(\Lambda) = \Omega_2$. In particular, we recall that $\Lambda \in \bar{\mathcal{U}}$ implies $|\mu| \geqslant (M^{(0)})^{\alpha/2}$. Moreover, we also point out that $a_n \geqslant 1$, so that conditions (69) are fulfilled for any n.

As we are dealing with a finite number of Diophantine conditions, then $\bar{\mathcal{E}}^{(n)}(R)$ is a set with a non-empty interior, for each $n \geqslant 0$. Therefore, at the limit $n \to +\infty$ it becomes a Cantor set,

$$\bar{\mathcal{E}}^{(\infty)} := \bigcap_{n\geqslant 0} \bar{\mathcal{E}}^{(n)}. \tag{82}$$

We point out that, *a priori*, we cannot guarantee that $\bar{\mathcal{E}}^{(\infty)}$ is non-empty. Moreover, we also recall that we are only interested in *real* basic frequencies, but that $\bar{\mathcal{E}}^{(\infty)}$ can be a complex set. These two topics are discussed in section 5.12.

Our purpose now is to ensure that if $M^{(0)}$ is small enough—i.e. if R is small enough—then all the requirements needed to apply lemma 5.4 to $H^{(n)}$ are fulfilled for any $\Lambda \in \bar{\mathcal{E}}^{(n)}$ and $n \geqslant 0$. We begin by assuming *a priori* that we can iterate indefinitely. If this were possible, using (78) and (80) we establish the following expression for $\bar{M}^{(n)}$:

$$\bar{M}^{(n)} = \frac{(\bar{\delta}^{(0)})^{4\tau+6}(M^{(0)})^{19\alpha}}{2\bar{\kappa}} 2^{-(n+1)(4\tau+6)}(\bar{\kappa}^{(0)})^{2^n}, \tag{83}$$

where

$$\bar{\kappa}^{(0)} := \frac{2^{4\tau+7}\bar{\kappa}(M^{(0)})^{1-19\alpha}}{(\bar{\delta}^{(0)})^{4\tau+6}}. \tag{84}$$

We point out that, as $0 < \alpha < 1/19$, we can make $\bar{\kappa}^{(0)}$ as small as required by simply taking R small enough. In particular, if we suppose $\bar{\kappa}^{(0)} \leqslant 1/2$, then the size of the 'error term' goes to zero with the step. The next consequence is that the inductive condition (70), formulated at the nth step, now reads as

$$2^{-(n+2)(2\tau+3)-1}(\bar{\delta}^{(0)})^{2\tau+3}(M^{(0)})^{5\alpha}(\bar{\kappa}^{(0)})^{2^n} \leqslant 1,$$

and clearly holds if $\bar{\kappa}^{(0)} \leqslant 1/2$ and $M^{(0)}$ is small enough.

Moreover, using (83) we can also give the explicit expression of the value $\bar{N} = \bar{N}^{(n)}(R)$ that we select for the ultra-violet cut-off. Thus, from condition (79) it is natural to take $\bar{N}^{(n)} := \lfloor \hat{N}^{(n)} \rfloor + 1$, with

$$\hat{N}^{(n)} := -\frac{1}{\delta^{(n)}} \log\left(\frac{\bar{\kappa}(\bar{M}^{(n)})}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}\right) = \frac{2^{2n}}{\bar{\delta}^{(0)}} \log\left(\frac{1}{\bar{\kappa}^{(0)}}\right) + \frac{2^n(4\tau+7)}{\bar{\delta}^{(0)}} \log(2). \tag{85}$$

From this definition one can clearly conclude that $\lim_{n\to+\infty} \bar{N}^{(n)} = +\infty$. In addition, if we also assume $\bar{\kappa}^{(0)} \leqslant 2^{-4\tau-7}\mathrm{e}^{-\bar{\delta}^{(0)}}$, then $\hat{N}^{(n+1)} \geqslant 3\hat{N}^{(n)} + 3$, and hence the iterative condition $\bar{N}^{(n+1)} > 3\bar{N}^{(n)}$ is also fulfilled. Finally, to simplify the control of $\bar{N}^{(n)}$ we observe that

$$\frac{2^{2n}}{\bar{\delta}^{(0)}} \log\left(\frac{1}{\bar{\kappa}^{(0)}}\right) \leqslant \bar{N}^{(n)} \leqslant \frac{2^{2n+1}}{\bar{\delta}^{(0)}} \log\left(\frac{1}{\bar{\kappa}^{(0)}}\right). \tag{86}$$

To finish ensuring the inductive applicability of the iterative lemma, we have to guarantee that the size of $\Omega_1^{(n)}$, $\mathcal{C}^{(n)}$, $\tilde{H}^{(n)}$ and $(\langle \bar{\mathcal{A}}^{(n)} \rangle_\theta)^{-1}$ is controlled, for each $n \geqslant 0$, as required in the statement. We do not plan to give full details and we only illustrate this process in terms of $(\langle \bar{\mathcal{A}}^{(n)} \rangle_\theta)^{-1}$, which turns out to be the term giving worst estimates. First, we recall that we have bound (68) for $(\langle \bar{\mathcal{A}}^{(0)} \rangle_\theta)^{-1}$. Moreover, the iterative application of the lemma gives

$$|(\langle \bar{\mathcal{A}}^{(n+1)} \rangle_\theta)^{-1} - (\langle \bar{\mathcal{A}}^{(n)} \rangle_\theta)^{-1}|_{\bar{\mathcal{E}}^{(n)}} \leqslant \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+3}(M^{(0)})^{29\alpha/2}}.$$

Then, it is natural to study the convergence of the following sum,

$$\sum_{n=0}^{\infty} \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+3}(M^{(0)})^{29\alpha/2}} = \sum_{n=0}^{\infty} (\bar{\delta}^{(0)})^{2\tau+3}(M^{(0)})^{9\alpha/2}\frac{(\bar{\kappa}^{(0)})^{2^n}}{2^{(n+2)(2\tau+3)+1}} \leqslant 2\bar{\kappa}\frac{(M^{(0)})^{1-29\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \quad (87)$$

where we have used that $2^n \geqslant n+1$ to bound the sum in terms of a geometrical progression of ratio $2^{-(2\tau+3)}\bar{\kappa}^{(0)} \leqslant 1/2$. By performing similar computations for the other terms, we obtain, for any $n \geqslant 1$,

$$|(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}^{(0)}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}^{(n-1)}} \leqslant 2\bar{\kappa}\frac{(M^{(0)})^{1-29\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \qquad |\Omega_1^{(n)} - \Omega_1^{(0)}|_{\bar{\mathcal{E}}^{(n-1)}} \leqslant 2\bar{\kappa}\frac{(M^{(0)})^{1-13\alpha/2}}{(\bar{\delta}^{(0)})^{\tau}},$$

$$|\tilde{H}^{(n)} - \tilde{H}^{(0)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},\bar{R}^{(n)}} \leqslant 2\bar{\kappa}\frac{(M^{(0)})^{1-19\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \qquad |\mathcal{C}^{(n)} - \mathcal{C}^{(0)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},0} \leqslant 2\bar{\kappa}\frac{(M^{(0)})^{1-27\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}.$$

We also have the direct bound $|H^{(n)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},\bar{R}^{(n)}} \leqslant \kappa$ (whenever it only involves compositions of functions). Then the estimates on $\tilde{H}^{(n)}$, $(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1}$ and $\mathcal{C}^{(n)}$ needed in the statement of lemma 5.4 hold if $\bar{\kappa}^{(0)} \leqslant \kappa$ (use definition of $\bar{\kappa}^{(0)}$ and bounds on the zero stage in (66), (67) and (68)). Of course, if we take $n \to +\infty$, then the same bounds hold for the limit Hamiltonian $H^{(\infty)}$ in (37).

5.10. *Convergence of the change of variables*

To finish the proof of the convergence of the iterative scheme, it only remains to check the convergence of the composition of the sequence of canonical transformations $\{\Psi_\Lambda^{(n)}\}_{n\geqslant 0}$. Concretely, we introduce $\tilde{\Psi}^{(n)} = \tilde{\Psi}_\Lambda^{(n)}$ defined as

$$\tilde{\Psi}^{(n)} := \Psi^{(0)} \circ \cdots \circ \Psi^{(n)}, \quad (88)$$

and we are going to prove that, for any $\Lambda \in \bar{\mathcal{E}}^{(\infty)}$, there exists $\tilde{\Psi}^{(\infty)} = \lim_{n\to+\infty} \tilde{\Psi}^{(n)}$, giving an analytic canonical transformation defined as $\tilde{\Psi}^{(\infty)} : \mathcal{D}_{2,1}(\rho^{(\infty)}, R^{(\infty)}) \to \mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$.

Remark 5.7. Of course, the dependence of $\tilde{\Psi}^{(\infty)}$ on $\Lambda \in \bar{\mathcal{E}}^{(\infty)}$ is no longer analytic but, as we are going to discuss in section 5.14, this transformation admits a Whitney-C^∞ extension. Moreover, $\tilde{\Psi}^{(\infty)}$ is not real analytic but, as discussed in remark 5.4, it can be *realified* (see section 5.13 for details).

To prove the convergence of $\tilde{\Psi}^{(\infty)}$ and to bound it we use lemma A.5. First, we note that from the iterative application of lemma 5.4 we have that $\Psi^{(n)} : \mathcal{D}_{2,1}(\bar{\rho}^{(n+1)}, \bar{R}^{(n+1)}) \to \mathcal{D}_{2,1}(\bar{\rho}^{(n)}, \bar{R}^{(n)})$, with

$$|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} \leqslant \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+1}(M^{(0)})^{19\alpha/2}} \leqslant (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{19\alpha/2}2^{-(n+2)(2\tau+5)+3}(\bar{\kappa}^{(0)})^{2^n},$$

$$|\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} \leqslant \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+2}(M^{(0)})^{15\alpha/2}} \leqslant (\bar{\delta}^{(0)})^{2\tau+4}(M^{(0)})^{23\alpha/2}2^{-(n+2)(2\tau+4)+1}(\bar{\kappa}^{(0)})^{2^n},$$

$$|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} \leqslant \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+5}(M^{(0)})^{21\alpha/2}} \leqslant (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{21\alpha/2}2^{-(n+2)(2\tau+5)+3}(\bar{\kappa}^{(0)})^{2^n}.$$

Then, according to lemma A.5, we have to consider the sum with respect to n of each of these bounds. Indeed (compare (87)),

$$\sum_{n=0}^{+\infty} |\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} \leqslant \frac{2\bar{\kappa}}{(\bar{\delta}^{(0)})^{2\tau+1}}(M^{(0)})^{1-19\alpha/2} := A. \quad (89)$$

Similarly, from the sum of the bounds on $\mathcal{I}^{(n)}$ and $\mathcal{Z}^{(n)}$ we can define, respectively,

$$B := \frac{2\bar{\kappa}}{(\bar{\delta}^{(0)})^{2\tau+2}}(M^{(0)})^{1-15\alpha/2}, \qquad C := \frac{2\bar{\kappa}}{(\bar{\delta}^{(0)})^{2\tau+1}}(M^{(0)})^{1-17\alpha/2}. \tag{90}$$

Next, we introduce $\bar{\rho}_-^{(n)} := \bar{\rho}^{(n)} - \bar{\delta}^{(0)}$ and $\bar{R}_-^{(n)} := \bar{R}^{(n)}\exp(-\bar{\delta}^{(0)})$. It is clear that $\lim_{n\to+\infty}(\bar{\rho}_-^{(n)}, \bar{R}_-^{(n)}) = (\rho^{(\infty)}, R^{(\infty)})$ (see (77) and (78)). Moreover, if we proceed analogously as in (76) and use condition $\bar{\kappa}^{(0)} \leqslant 1/2$ (recall also that $R^{(n)} \geqslant R^{(\infty)} \geqslant (M^{(0)})^{\alpha}/2$), we also have

$$|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} \leqslant \bar{\rho}_-^{(n)} - \bar{\rho}_-^{(n+1)}, \qquad |\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} \leqslant (\bar{R}_-^{(n)})^2 - (\bar{R}_-^{(n+1)})^2,$$
$$|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} \leqslant \bar{R}_-^{(n)} - \bar{R}_-^{(n+1)}.$$

Then, under the above conditions we guarantee the applicability of lemma A.5. Consequently, from point (ii) of the lemma, we have the following bounds for the components of $\tilde{\Psi}^{(\infty)} - \mathrm{Id}$ (see (5)):

$$|\tilde{\Theta}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leqslant A, \qquad |\tilde{\mathcal{I}}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leqslant B, \qquad |\tilde{\mathcal{Z}}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leqslant C. \tag{91}$$

Finally, we use point (iii) of lemma A.5 to bound the difference between the components of $\tilde{\Psi}^{(n)}$ and $\tilde{\Psi}^{(n-1)}$. Thus, we define for each $n \geqslant 1$,

$$\Pi_n := \frac{1}{\bar{\delta}^{(0)}}\left(\frac{2|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}}}{\exp(1)} + \frac{2|\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}}}{(R^{(\infty)})^2} + \frac{4|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}}}{R^{(\infty)}}\right)$$
$$\leqslant (\bar{\delta}^{(0)})^{2\tau+3}(M^{(0)})^{19\alpha/2}2^{-(n+2)(2\tau+4)+5}(\bar{\kappa}^{(0)})^{2^n},$$

and we have, for R small enough,

$$\begin{aligned}|\tilde{\Theta}^{(n)} - \tilde{\Theta}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} &\leqslant |\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}^{(n+1)},\bar{R}^{(n+1)}} + A\Pi_n \\ &\leqslant (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{19\alpha/2}2^{-(n+2)(2\tau+4)+1}(\bar{\kappa}^{(0)})^{2^n}.\end{aligned}$$

Similarly, we establish analogous bounds for the other components:

$$\begin{aligned}|\tilde{\mathcal{I}}^{(n)} - \tilde{\mathcal{I}}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} &\leqslant (\bar{\delta}^{(0)})^{2\tau+4}(M^{(0)})^{23\alpha/2}2^{-(n+2)(2\tau+4)+2}(\bar{\kappa}^{(0)})^{2^n}, \\ |\tilde{\mathcal{Z}}^{(n)} - \tilde{\mathcal{Z}}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},\bar{R}_-^{(n+1)}} &\leqslant (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{21\alpha/2}2^{-(n+2)(2\tau+4)+1}(\bar{\kappa}^{(0)})^{2^n}.\end{aligned}$$

5.11. Bound on the Lipschitz constant of $\Omega_1^{(n)}$

Once we have proved the full convergence of the KAM iterative process, actually we have that, for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2$ (see (82)), there is an invariant torus of the non-integrable Hamiltonian system $\mathcal{H}$ in (10), with normal frequency μ and intrinsic frequencies $\Omega^{(\infty)}(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$, where $\Omega_1^{(\infty)} = \lim_{n\to+\infty}\Omega_1^{(n)}$. However, the mere convergence of the sequence $\Omega_1^{(n)}$ is not enough in order to build measure estimates along the iterative process. We also require some additional information about the Lipschitz constant of $\Omega_1^{(n)}$, which can be derived from the control of their partial derivatives. As we know, by construction, that $\Omega_1^{(n)}(\Lambda)$ depends analytically on $\Lambda \in \bar{\mathcal{E}}^{(n-1)}$ (see (81)), this process can be done by means of Cauchy estimates. To do that, we need to control the distance to the boundary of the points inside the set $\bar{\mathcal{E}}^{(n)}$ of 'admissible' basic frequencies at the nth step.

For this purpose, we introduce the following sequence of sets. First, we define $\hat{\mathcal{U}} = \hat{\mathcal{U}}(R)$ as (compare $\bar{\mathcal{U}}$ in (64))

$$\begin{aligned}\hat{\mathcal{U}} := \big\{\Lambda \in \mathbb{C}^2 :\ & 2(M^{(0)})^{\alpha/2} \leqslant \mathrm{Re}\,\mu,\ |\mathrm{Im}\,\mu| \leqslant (M^{(0)})^{\alpha/2},\ |\mu| \leqslant c_2R - (M^{(0)})^{\alpha/2}, \\ & |\Omega_2 - \omega_2| \leqslant c_2R - (M^{(0)})^{\alpha/2}, |\mathrm{Im}\,\Omega_2| \leqslant (M^{(0)})^{\alpha/2}, |h_1(\Lambda)| \geqslant 2(M^{(0)})^{\alpha/2}\big\},\end{aligned} \tag{92}$$

and thus, in analogy to (81), we set $\hat{\mathcal{E}}^{(-1)} = \hat{\mathcal{U}}$ and, for any $n \geqslant 0$,

$$\hat{\mathcal{E}}^{(n)} := \{\Lambda \in \hat{\mathcal{E}}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geqslant b_n (M^{(0)})^{\alpha/2} |k|_1^{-\tau}, \quad 0 < |k|_1 < 2\bar{N}^{(n)}, \ \ell \in \{0,1,2\}\}, \tag{93}$$

now with $b_n = 1 + 2^{-n+1}$. It is clear that, by construction, we always have $\hat{\mathcal{E}}^{(n)} \subset \bar{\mathcal{E}}^{(n)}$ (observe that $b_n > a_n$). After that, we introduce the sequence of positive numbers $\nu^{(n)} = \nu^{(n)}(R) > 0$ given by $\nu^{(-1)} := (M^{(0)})^{\alpha/2}/3$ and

$$\nu^{(n)} := 2^{-n-\tau-3} \frac{(M^{(0)})^{\alpha/2}}{(\bar{N}^{(n)})^{\tau+1}}, \qquad n \geqslant 0. \tag{94}$$

For further use, we observe that (86) implies the lower bound

$$\nu^{(n)} \geqslant \frac{(\bar{\delta}^{(0)})^{\tau+1} (M^{(0)})^{\alpha/2}}{2^{2\tau+4}} \left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{-\tau-1} 2^{-n(2\tau+3)}, \qquad n \geqslant 0. \tag{95}$$

Our objective is to show that if R is small enough, then $\hat{\mathcal{E}}^{(n)} + 3\nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$, for each $n \geqslant -1$ (see (9)). Using this inclusion we can control the partial derivatives of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$ and its Lipschitz constant in $\hat{\mathcal{E}}^{(n)} + \nu^{(n)}$.

We start with $n = -1$. In this case we have to prove that $\hat{\mathcal{U}} + (M^{(0)})^{\alpha/2} \subset \bar{\mathcal{U}}$. This means that if we take an arbitrary $\Lambda \in \hat{\mathcal{U}}$ and Λ' is such that $|\Lambda' - \Lambda| \leqslant (M^{(0)})^{\alpha/2}$, then $\Lambda' \in \bar{\mathcal{U}}$. This is clear from the definition of both sets, except for what concerns the lower bound on $|h_1(\Lambda)|$. But using lemma 5.2 we have

$$\begin{aligned} |h_1(\Lambda')| &\geqslant |h_1(\Lambda)| - |h_1(\Lambda') - h_1(\Lambda)| \\ &\geqslant 2(M^{(0)})^{\alpha/2} - \frac{2c_2R}{a}|\mu' - \mu| - \frac{8c_1R}{a}|\Omega_2' - \Omega_2| \geqslant (M^{(0)})^{\alpha/2}, \end{aligned}$$

provided that R is small enough.

After that we proceed by induction with respect to n. Concretely, we want to prove that, for any $n \geqslant 0$, the following properties hold:

$$|\partial_\mu \Omega_1^{(n)}|_{\hat{\mathcal{E}}^{(n-1)}+2\nu^{(n-1)}} \leqslant 2c_3R, \qquad |\partial_{\Omega_2} \Omega_1^{(n)}|_{\hat{\mathcal{E}}^{(n-1)}+2\nu^{(n-1)}} \leqslant 2c_3R, \tag{96}$$

$$\hat{\mathcal{E}}^{(n-1)} + 3\nu^{(n-1)} \subset \bar{\mathcal{E}}^{(n-1)}, \qquad \mathrm{Lip}_{\hat{\mathcal{E}}^{(n-1)}+\nu^{(n-1)}}(\Omega_1^{(n)}) \leqslant 4c_3R. \tag{97}$$

From the above discussions and lemma 5.3 it is clear that (96) and (97) hold when $n = 0$ (recall that $\hat{\mathcal{U}} \subset \bar{\mathcal{U}} \subset \mathcal{U}$). Let us suppose them to be true for a given $n \geqslant 0$ and we verify them for the next case.

We first prove that $\hat{\mathcal{E}}^{(n)} + 3\nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$. Let $\Lambda \in \hat{\mathcal{E}}^{(n)}$ be fixed and take Λ' such that $|\Lambda' - \Lambda| \leqslant 3\nu^{(n)}$. As the set $\hat{\mathcal{E}}^{(n)}$ is defined from $\hat{\mathcal{E}}^{(n-1)}$ and $3\nu^{(n)} \leqslant \nu^{(n-1)}$, it is clear that both $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n-1)} + \nu^{(n-1)}$, so that we can use the Lipschitz estimate (97) on $\Omega_1^{(n)}$. To check that $\Lambda' \in \bar{\mathcal{E}}^{(n)}$ we compute, for any $k \in \mathbb{Z}^2$ with $0 < |k|_1 < 2\bar{N}^{(n)}$ and $\ell \in \{0,1,2\}$ (recall that $\Omega^{(n)}(\Lambda) = (\Omega_1^{(n)}(\Lambda), \Omega_2)$),

$$\begin{aligned} &|\langle k, \Omega^{(n)}(\Lambda')\rangle + \ell\mu'| \\ &\quad \geqslant |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| - |k_1||\Omega_1^{(n)}(\Lambda') - \Omega_1^{(n)}(\Lambda)| - |k_2||\Omega_2' - \Omega_2| - |\ell||\mu' - \mu| \\ &\quad \geqslant b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau} - 4c_3R|k_1||\Lambda' - \Lambda| - |k_2||\Omega_2' - \Omega_2| - 2|\mu' - \mu| \\ &\quad \geqslant \left(b_n(M^{(0)})^{\alpha/2} - 4\nu^{(n)}(2\bar{N}^{(n)})^{\tau+1}\right)|k|_1^{-\tau} \\ &\quad \geqslant a_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \end{aligned}$$

if R is small enough (condition depending only on c_3). Here, we use definition (94) and $b_n - a_n = 2^{-n}$.

The following step is to control the partial derivatives of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$. From the iterative application of lemma 5.4, we can define the analytic function $\Omega_1^{(j+1)}$ in the complex set $\bar{\mathcal{E}}^{(j)}$, with

$$|\Omega_1^{(j+1)} - \Omega_1^{(j)}|_{\bar{\mathcal{E}}^{(j)}} \leqslant \frac{\bar{\kappa}\bar{M}^{(j)}}{(\bar{\delta}^{(j)})^{\tau}(M^{(0)})^{13\alpha/2}}, \qquad j \geqslant 0. \tag{98}$$

Using standard Cauchy estimates and the inductive inclusion $\hat{\mathcal{E}}^{(j)} + 3\nu^{(j)} \subset \bar{\mathcal{E}}^{(j)}$, for $j = 0, \ldots, n$, we obtain (in order to bound the sum below compare (87) and recall (84))

$$\begin{aligned} |\partial_\mu(\Omega_1^{(n+1)} - \Omega_1^{(0)})|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} &\leqslant \sum_{j=0}^{n} \frac{|\Omega_1^{(j+1)} - \Omega_1^{(j)}|_{\bar{\mathcal{E}}^{(j)}}}{\nu^{(j)}} \\ &\leqslant \sum_{j=0}^{\infty} 2^{-(j+2)(\tau+3)+3}(\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{12\alpha}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}(\bar{\kappa}^{(0)})^{2^j} \\ &\leqslant 2^{2\tau+5}\bar{\kappa}\frac{(M^{(0)})^{1-7\alpha}}{(\bar{\delta}^{(0)})^{2\tau+1}}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}. \end{aligned} \tag{99}$$

If we require R small enough so that (99) is bounded by $c_3 R$, then we have (see lemma 5.3)

$$|\partial_\mu \Omega_1^{(n+1)}|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} \leqslant |\partial_\mu \Omega_1^{(0)}|_{\mathcal{U}} + |\partial_\mu(\Omega_1^{(n+1)} - \Omega_1^{(0)})|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} \leqslant 2c_3 R. \tag{100}$$

The same works for the other partial derivative, $\partial_{\Omega_2}(\Omega_1^{(n+1)})$, so that (96) holds for any $n \geqslant 0$.

Finally, we discuss the Lipschitz constant of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + \nu^{(n)}$. Clearly, this Lipschitz constant can be locally bounded in terms of the partial derivatives by $4c_3 R$. But it only holds for points such that their union segment is contained inside the domain $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$. However, the sequence of Diophantine conditions we have imposed on the original (convex) domain of basic frequencies $\mathcal{U}$ (see (62)) has created multiple holes in the set $\hat{\mathcal{E}}^{(n)}$. Thus, we can only guarantee linear connectivity inside $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$ for points $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n)} + \nu^{(n)}$ so that $|\Lambda' - \Lambda| \leqslant \nu^{(n)}$. But if we pick up points so that $|\Lambda' - \Lambda| \geqslant \nu^{(n)}$, we can alternatively control their 'Lipschitz constant' using the norm of the function and the lower bound on their separation. Concretely,

$$|(\Omega_1^{(n+1)}(\Lambda') - \Omega_1^{(n)}(\Lambda')) - (\Omega_1^{(n+1)}(\Lambda) - \Omega_1^{(n)}(\Lambda))| \leqslant \frac{2|\Omega_1^{(n+1)} - \Omega_1^{(n)}|_{\bar{\mathcal{E}}^{(n)}}}{\nu^{(n)}}|\Lambda' - \Lambda|,$$

which is the same bound that we obtain in the 'local case' using Cauchy estimates. Then, taking into account this methodology for controlling the 'Lipschitz constant' for 'separated points', we can adapt the procedure used in (99) and (100) in order to control $|\Omega_1^{(n+1)}(\Lambda') - \Omega_1^{(n+1)}(\Lambda)|$ for $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n)} + \nu^{(n)}$, independently of their distance. We leave the details to the reader.

5.12. Measure estimates

Now, we have at hand all the ingredients needed to discuss the Lebesgue measure of the set of basic frequencies giving an invariant torus linked to the Hopf bifurcation. But, as we are only interested in *real* basic frequencies, we first introduce the following sets:

$$\hat{\mathcal{E}}^{(\infty)}(R) := \bigcap_{n\geqslant 0} \hat{\mathcal{E}}^{(n)}(R), \qquad \mathcal{E}^{(\infty)}(R) := \hat{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2,$$

$$\mathcal{E}^{(n)}(R) := \hat{\mathcal{E}}^{(n)}(R) \cap \mathbb{R}^2, \qquad n \geqslant -1, \tag{101}$$

and (see (62))

$$\mathcal{V}(R) := \mathcal{U}(R) \cap (\mathbb{R}^+ \times \mathbb{R}) = \{\Lambda = (\mu, \Omega_2) \in \mathbb{R}^2 : 0 < \mu \leqslant c_2 R,\ |\Omega_2 - \omega_2| \leqslant c_2 R\}. \tag{102}$$

In a few words, $\mathcal{V} = \mathcal{V}(R)$ is the initial set of real basic frequencies in which we look for invariant tori and $\mathcal{E}^{(\infty)} = \mathcal{E}^{(\infty)}(R)$ is the corresponding subset in which we have proved the convergence of the KAM process. To size up the holes between invariant tori, we have to control the Lebesgue measure meas $(\mathcal{V} \setminus \mathcal{E}^{(\infty)})$. For this purpose we write

$$\mathcal{V} \setminus \mathcal{E}^{(\infty)} = \left(\mathcal{V} \setminus \mathcal{E}^{(-1)}\right) \bigcup \left(\bigcup_{n \geqslant 0} (\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)}) \right). \tag{103}$$

We start by controlling meas $(\mathcal{V} \setminus \mathcal{E}^{(-1)})$. From the definition of $\hat{\mathcal{E}}^{(-1)} = \hat{\mathcal{U}}$ (see (92)), we have

$$\mathcal{E}^{(-1)} = \left\{ \Lambda \in \mathbb{R}^2 \ : 2(M^{(0)})^{\alpha/2} \leqslant \mu \leqslant c_2 R - (M^{(0)})^{\alpha/2},\ |\Omega_2 - \omega_2| \leqslant c_2 R - (M^{(0)})^{\alpha/2}, \right.$$
$$\left. |h_1(\Lambda)| \geqslant 2(M^{(0)})^{\alpha/2} \right\}.$$

So, if we get rid of the lower bound $|h_1(\Lambda)| \geqslant 2(M^{(0)})^{\alpha/2}$, then we clearly obtain an estimate of $\mathcal{O}((M^{(0)})^{\alpha/2})$ for this measure. Unfortunately, this estimate is worsened when adding the condition on h_1. We recall that the vector function $h^{(R)} = (h_1, h_2)$, depending on R, and so on the selected normal form order, has been introduced in lemma 5.2. Concretely, h denotes the inverse of the transformation $\zeta = (\xi, \eta) \mapsto \Lambda = (\mu, \Omega_2)$, defined by the parametrization in terms of ζ of the 2D-bifurcated invariant tori of the normal form (see theorem 4.2), i.e. $\xi = h_1(\Lambda)$. For further use, and to prevent possible confusion with the basic frequencies themselves, we denote by $\Upsilon = \Upsilon^{(R)}(\zeta)$ the vector function having as components $\Upsilon_1 = \mu(\zeta)$ and $\Upsilon_2 = \Omega_2(\zeta)$, defined by the R-dependent parametrizations (24), (27) and (28). Therefore, due to the square root of the definition of μ in (28), we have to be very careful to select the domain for the vector-function Υ.

According to lemma 5.1, the function $\mathbb{I} = \mathbb{I}^{(R)}$ is analytic in the (complex) domain $\Gamma = \Gamma(R)$ (see (60)) and so is $\Omega_2(\zeta)$ (see (24)). Thus, it is natural to consider the following (real) domain for Υ (see remark 4.4 for more details):

$$\Gamma^* = \Gamma^*(R) := \{\zeta \in \Gamma \cap \mathbb{R}^2 : 4\eta^2 + 2a\xi + 2\xi \partial^2_{1,1} Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) > 0\}.$$

Moreover, we consider the auxiliary sets $A = A(R)$ and $B = B(R)$ given by

$$A := \{\Lambda \in \mathbb{R}^2 : 0 < \mu \leqslant c_2 R,\ |\Omega_2 - \omega_2| \leqslant c_2 R,\ |h_1(\Lambda)| \leqslant 2(M^{(0)})^{\alpha/2}\},$$
$$B := \{\zeta \in \Gamma^* : |\xi| \leqslant 2(M^{(0)})^{\alpha/2},\ |\eta| \leqslant c_1 R\},$$

where c_1 has been introduced in lemma 5.1. We stress that the restriction $\zeta \in \Gamma^*$ in the definition of B also implies that $\Upsilon_1(\zeta) > 0$. It is clear that by bounding meas (A) we control the effect of the lower bound $|h_1(\Lambda)| \geqslant 2(M^{(0)})^{\alpha/2}$ on meas $(\mathcal{V} \setminus \mathcal{E}^{(-1)})$. Then, the important thing is that $A \subset \Upsilon(B)$, so that bounding meas (A) can be done by bounding the Jacobian of Υ in B. From expressions (24), (27), (28), the bounds in (59) on the partial derivatives of $Z = Z^{(R)}$ and of lemma 5.1 on $\mathbb{I} = \mathbb{I}^{(R)}$, we have that, for any $\zeta \in \Gamma^*$ and R small enough,

$$|\partial_\xi \Upsilon_1(\zeta)| = \frac{\partial_\xi(\Upsilon_1^2(\zeta))}{2\Upsilon_1(\zeta)}, \qquad |\partial_\eta \Upsilon_1(\zeta)| \leqslant \frac{8c_1 R}{\Upsilon_1(\zeta)},$$
$$|\partial_\xi \Upsilon_2(\zeta)| \leqslant 2|e| + 4\frac{a|f|}{|d|}, \qquad |\partial_\eta \Upsilon_2(\zeta)| \leqslant 2,$$

where we take special care in making explicit the effect on the derivatives of the square root defining $\Upsilon_1 = \mu$. Here we can bound the Jacobian of Υ by

$$\left|\det(\partial_\zeta \Upsilon)(\zeta)\right| \leqslant \frac{3}{2} \frac{\partial_\xi(\Upsilon_1^2(\zeta))}{\Upsilon_1(\zeta)},$$

where the key point is that $\partial_\xi(\Upsilon_1^2(\zeta)) \geqslant a > 0$, if R is small enough. Then we have

$$\text{meas}\,(A) = \iint_A \mathrm{d}\mu\,\mathrm{d}\Omega_2 \leqslant \iint_{\Upsilon(B)} \mathrm{d}\mu\,\mathrm{d}\Omega_2 \leqslant \iint_B \frac{3}{2}\frac{\partial_\xi(\Upsilon_1^2(\zeta))}{\Upsilon_1(\zeta)}\mathrm{d}\xi\,\mathrm{d}\eta.$$

The integral with respect to ξ on the right-hand side can be computed explicitly, giving an expression of the form $3(\Upsilon_1(\xi',\eta) - \Upsilon_1(\xi,\eta))$, for certain $\xi = \xi(\eta)$ and $\xi' = \xi'(\eta)$. In its turn, this expression has to be integrated with respect to η. In order to avoid the square root defining Υ_1 we recall the Hölder bound

$$|\sqrt{x} - \sqrt{y}| \leqslant |x-y|^{1/2}, \qquad x, y > 0.$$

Hence we have, for small R and $(\xi,\eta), (\xi',\eta) \in B$,

$$\begin{aligned}|\Upsilon_1(\xi',\eta) - \Upsilon_1(\xi,\eta)| &\leqslant |\Upsilon_1^2(\xi',\eta) - \Upsilon_1^2(\xi,\eta)|^{1/2}\\ &= |2a(\xi'-\xi) + 2\xi'\partial_{1,1}^2 Z(\xi', \mathbb{I}(\xi',\eta), \xi'\eta) - 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi,\eta), \xi\eta)|^{1/2}\\ &\leqslant 4\sqrt{a}(M^{(0)})^{\alpha/4},\end{aligned}$$

which finally gives

$$\text{meas}\,(A) \leqslant 24\sqrt{a}c_1 R (M^{(0)})^{\alpha/4}.$$

To obtain this estimate, apart from the explicit expression of $\Upsilon_1 = \mu$, we use the bounds on the partial derivatives of Z and the definition of the set B.

As a conclusion, we have established the following bound

$$\text{meas}\,(\mathcal{V} \setminus \mathcal{E}^{(-1)}) \leqslant c_4 (M^{(0)})^{\alpha/4}, \tag{104}$$

for certain $c_4 > 0$ independent of R.

The next step is to bound meas $(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})$ for any $n \geqslant 0$. This is a standard process (compare for instance [28]) which only requires suitable transversality conditions in order to deal with the Diophantine conditions defining the sets at hand. In the present context, these transversality conditions are immediate from the bifurcation scenario we are discussing and our 'adequate' choice of the basic frequencies (see (31)). We consider the following decomposition,

$$\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)} = \bigcup_{\ell\in\{0,1,2\}} \bigcup_{0<|k|_1<2\bar{N}^{(n)}} \mathcal{R}_{\ell,k}^{(n)},$$

where $\mathcal{R}_{\ell,k}^{(n)}$ contains the basic frequencies for which one of the Diophantine conditions defining $\mathcal{E}^{(n)}$ fails (see (93) and (101)). Concretely,

$$\mathcal{R}_{\ell,k}^{(n)} = \{\Lambda \in \mathcal{E}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| < b_n (M^{(0)})^{\alpha/2}|k|_1^{-\tau}\}.$$

To control meas $(\mathcal{R}_{\ell,k}^{(n)})$, we first suppose that $(\ell, k_2) \neq (0,0)$ and take a couple $\Lambda, \Lambda' \in \mathcal{R}_{\ell,k}^{(n)}$ such that $\Lambda - \Lambda'$ is parallel to the vector (ℓ, k_2). Then we have

$$\begin{aligned}|\Lambda - \Lambda'|_2 &= \frac{1}{|(\ell,k_2)|_2}|\langle \Lambda - \Lambda', (\ell,k_2)\rangle| = \frac{1}{|(\ell,k_2)|_2}|(k_2\Omega_2 + \ell\mu) - (k_2\Omega_2' + \ell\mu')|\\ &\leqslant \frac{1}{|(\ell,k_2)|_2}\left(2b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + |k_1||\Omega_1^{(n)}(\Lambda) - \Omega_1^{(n)}(\Lambda')|\right)\\ &\leqslant \frac{1}{|(\ell,k_2)|_2}\left(6(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 4c_3 R|k_1||\Lambda - \Lambda'|\right),\end{aligned}$$

where we recall that $b_n \leqslant 3$ and that $\text{Lip}_{\hat{\mathcal{E}}^{(n-1)}}(\Omega_1^{(n)}) \leqslant 4c_3 R$ (see (97)). Thus, we obtain

$$\left(1 - 4c_3 R\frac{|k_1|}{|(\ell,k_2)|_2}\right)|\Lambda - \Lambda'| \leqslant 6(M^{(0)})^{\alpha/2}\frac{|k|_1^{-\tau}}{|(\ell,k_2)|_2}.$$

If we assume, for the moment, that $4c_3R|k_1|/|(\ell, k_2)|_2 \leqslant 1/2$, then we finally end with the estimate

$$|\Lambda - \Lambda'| \leqslant 12(M^{(0)})^{\alpha/2}\frac{|k|_1^{-\tau}}{|(\ell, k_2)|_2}. \tag{105}$$

To ensure this assumption, we study for which values of k_1 it could be $\mathcal{R}^{(n)}_{\ell,k} \neq \emptyset$. Thus, let us suppose that $\Lambda = (\mu, \Omega_2)$ belongs to $\mathcal{R}^{(n)}_{\ell,k}$. If we set R small enough so that $\max\{|\Omega_2|, \mu\} \leqslant 2|\omega_2|$, we have

$$|k_1||\Omega_1^{(n)}(\Lambda)| \leqslant |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| + |k_2||\Omega_2| + |\ell||\mu| \leqslant 3(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 4|\omega_2||(\ell, k_2)|_2.$$

If we also assume R small enough such that $|\Omega_1^{(n)}(\Lambda)| \geqslant |\omega_1|/2$, then

$$|k_1| \leqslant \frac{6}{|\omega_1|}(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 8\frac{|\omega_2|}{|\omega_1|}|(\ell, k_2)|_2,$$

which clearly implies $4c_3R|k_1|/|(\ell, k_2)|_2 \leqslant 1/2$, for R small. Moreover, if $(\ell, k_2) \neq (0, 0)$, we also deduce, for small R,

$$|k|_1 = |k_1| + |k_2| \leqslant \frac{6}{|\omega_1|}(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + \left(1 + 8\frac{|\omega_2|}{|\omega_1|}\right)|(\ell, k_2)|_2 \leqslant 2\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)|(\ell, k_2)|_2.$$

From here, we can rewrite (105) as

$$|\Lambda - \Lambda'| \leqslant 24\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)(M^{(0)})^{\alpha/2}\frac{1}{|k|_1^{\tau+1}}.$$

Once we have bounded the width of $\mathcal{R}^{(n)}_{\ell,k}$, in the direction given by the vector (ℓ, k_2), then, by taking into account the diameter of the set $\mathcal{V}$ (see (102)), we obtain the following estimate for its measure,

$$\text{meas}\,(\mathcal{R}^{(n)}_{\ell,k}) \leqslant 24\sqrt{5}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2R(M^{(0)})^{\alpha/2}\frac{1}{|k|_1^{\tau+1}}. \tag{106}$$

It remains to control this measure when $(\ell, k_2) = (0, 0)$. However, these cases can be omitted because $\mathcal{R}^{(n)}_{0,(k_1,0)} = \emptyset$ if $k_1 \neq 0$. Indeed,

$$|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| = |k_1||\Omega_1^{(n)}(\Lambda)| \geqslant |\omega_1|/2.$$

Then, using decomposition (103), we have

$$\text{meas}\left(\bigcup_{n\geqslant 0}(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})\right) = \sum_{n=0}^{\infty}\text{meas}\,(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)}) \leqslant \sum_{n=0}^{\infty}\sum_{\ell\in\{0,1,2\}}\sum_{0<|k|_1<2\bar{N}^{(n)}}\text{meas}\,(\mathcal{R}^{(n)}_{\ell,k}).$$

Unfortunately, this expression diverges if we just use estimate (106), because it does not depend on the index n. Nevertheless, we will show below that, for any $(\ell, k) \in \{0, 1, 2\} \times (\mathbb{Z}^2 \setminus \{0\})$, there is at most one $n = n^*(|k|_1)$ so that $\mathcal{R}^{(n)}_{\ell,k}$ is non-empty. Assuming that this assertion is true, we have

$$\begin{aligned}\text{meas}\left(\bigcup_{n\geqslant 0}(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})\right) &\leqslant \sum_{\ell\in\{0,1,2\}}\sum_{0<|k|_1<2\bar{N}^{(n)}}\text{meas}\,(\mathcal{R}^{(n^*(|k|_1))}_{\ell,k}) \\ &\leqslant 288\sqrt{5}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2R(M^{(0)})^{\alpha/2}\sum_{j=1}^{\infty}\frac{1}{j^{\tau}} \\ &\leqslant 288\sqrt{5}\frac{\tau}{\tau - 1}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2R(M^{(0)})^{\alpha/2}, \end{aligned} \tag{107}$$

where we use that $\#\{k \in \mathbb{Z}^2 : |k|_1 = j\} = 4j$ and $\sum_{j=1}^{+\infty} j^{-\tau} \leqslant 1 + \int_1^{\infty} x^{-\tau}\,\mathrm{d}x$ (recall $\tau > 1$).

Let us prove the above assertion. To be precise, given a fixed $k \in \mathbb{Z}^2 \setminus \{0\}$, we denote by $n^* = n^*(|k|_1) \geqslant 0$ the first index so that $|k|_1 < 2\bar{N}^{(n^*)}$. Then, we are going to show that $\mathcal{R}_{\ell,k}^{(n)} = \emptyset$ for any $n \neq n^*(|k|_1)$ and $\ell \in \{0, 1, 2\}$. In a few words, this means that if n^* is the first index n for which the small divisor of order (ℓ, k) is taken into account in the definition of the set of valid basic frequencies $\mathcal{E}^{(n)}$ (see (93) and (101)), then the 'resonant zone' $\mathcal{R}_{\ell,k}^{(n^*)}$ determines completely the values of Λ for which the Diophantine condition of order (ℓ, k) can fail at any step. Thus, if the required Diophantine condition of order (ℓ, k) is fulfilled for certain Λ at the step $n^*(|k|_1)$, then this basic frequency cannot fall into a resonant zone $\mathcal{R}_{\ell,k}^{(n)}$ for any $n > n^*$. We prove this property as follows.

Let $n \geqslant 1$ and $\Lambda \in \mathcal{E}^{(n-1)}$. This means that $\Omega_1^{(n-1)}(\Lambda)$ verifies

$$|\langle k, \Omega^{(n-1)}(\Lambda)\rangle + \ell\mu| \geqslant b_{n-1}(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \quad 0 < |k|_1 < 2\bar{N}^{(n-1)}, \qquad \ell \in \{0, 1, 2\}.$$

Then, we want to show that the following Diophantine conditions on $\Omega_1^{(n)}(\Lambda)$ are verified automatically,

$$|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geqslant b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \qquad 0 < |k|_1 < 2\bar{N}^{(n-1)}, \quad \ell \in \{0, 1, 2\}. \tag{108}$$

Thus, bounds (108) imply that to define $\mathcal{E}^{(n)}$ we only have to worry about the Diophantine conditions on $\Omega_1^{(n)}(\Lambda)$ of order $2\bar{N}^{(n-1)} \leqslant |k|_1 < 2\bar{N}^{(n)}$. Indeed,

$$\begin{aligned}|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| &\geqslant |\langle k, \Omega^{(n-1)}(\Lambda)\rangle + \ell\mu| - |k|_1|\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}} \\ &\geqslant \left(b_{n-1} - \frac{(2\bar{N}^{(n-1)})^{\tau+1}}{(M^{(0)})^{\alpha/2}}|\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}}\right)(M^{(0)})^{\alpha/2}|k|_1^{-\tau}.\end{aligned}$$

On the other hand, using (83), (86), (98) and that $\bar{\kappa}^{(0)} \leqslant 1$, we have, for any $n \geqslant 1$,

$$\begin{aligned}\frac{(2\bar{N}^{(n-1)})^{\tau+1}}{(M^{(0)})^{\alpha/2}}|\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}} &\leqslant 2^{-n(4+\tau)-\tau-1}(\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{12\alpha}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}(\bar{\kappa}^{(0)})^{2^{n-1}} \\ &\leqslant 2^{-n+1} = b_{n-1} - b_n,\end{aligned}$$

provided that $M^{(0)}$ is small enough (i.e. R small enough). Hence, we conclude that (108) holds.

Finally, if we put estimate (104) together with (107), we obtain the measure estimates of (13)

$$\text{meas}\,(\mathcal{V}(R) \setminus \mathcal{E}^{(\infty)}(R)) \leqslant c_5(M^{(0)}(R))^{\alpha/4},$$

for certain $c_5 > 0$ independent of R.

5.13. Real invariant tori

Now, it is time to return to the original system of coordinates of the problem, and to discuss which of the tori we have obtained are *real* tori when expressed in such coordinates. We recall that we have settled $\mathcal{H}$ in (10) to be our initial Hamiltonian. This system is written in a canonical set of (real) coordinates, namely $(\theta, x_1, x_2, I, y_1, y_2)$, which have been introduced with the requirement that the normal variational equations of the critical periodic orbit are of constant coefficients. Later on, we have modified these original coordinates throughout the paper, according to the different steps of the proof of theorem 3.1. Let us summarize here this sequence of changes.

(i) We have applied to $\mathcal{H}$ the R-dependent normal form transformation $\hat{\Psi}^{(R)}$ given by theorem 4.1.

(ii) We have introduced action-angle-like coordinates to this (partially) normalized system through the (R-independent) change (19). This transformation is not properly a complexification, but we need $q > 0$ in order to have real tori.

(iii) We have considered the Λ-dependent coordinate change (32), which moves to the 'origin' the unperturbed 2D-bifurcated torus having a vector of basic frequencies Λ and arranges its variational equations. This transformation involves the complex 'diagonalizing' change (33) but, as we have discussed in section 5.2, all the invariant tori we compute are real when written in the action-angle coordinates (19). Therefore, we only have to worry about the condition $q > 0$.

(iv) Finally, we have performed the KAM process. Thus, we have to compose all these coordinate changes with the limit KAM transformation $\tilde{\Psi}^{(\infty)} = \tilde{\Psi}^{(\infty)}_{\Lambda}(\theta_1, \theta_2, x, I_1, I_2, y)$ (see section 5.10), which is well defined for any $\Lambda \in \mathcal{E}^{(\infty)}$ (see (101)).

The most important property of the KAM transformation $\tilde{\Psi}^{(\infty)} = \mathrm{Id}+(\tilde{\Theta}^{(\infty)}, \tilde{\mathcal{X}}^{(\infty)}, \tilde{\mathcal{I}}^{(\infty)}, \tilde{\mathcal{Y}}^{(\infty)})$ is that if we set $x = y = I_1 = I_2 = 0$, then we obtain the parametrization, as a function of $\theta = (\theta_1, \theta_2)$, of the corresponding Λ-invariant torus of the 'full' system $H^{(0)}_{\Lambda}$ in (34). After composition of this parametrization with the transformations described above, we obtain the invariant tori of the initial system (written in the original variables). If we want to detect which of these tori are real, we have to study the sign of the variable q evaluated on any of them. Thus, abusing notation, we denote by $q^{(\infty)}(\theta, \Lambda)$ this coordinate function, which is obtained by replacing in (32) the variables (x, y, I_1, I_2) by the parametrization of the tori. Indeed,

$$q^{(\infty)}(\theta, \Lambda) := \xi + \tilde{\mathcal{X}}^{(\infty)} - \frac{\xi}{\lambda_+}\tilde{\mathcal{Y}}^{(\infty)} - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}})\tilde{\mathcal{I}}^{(\infty)}_1 - \frac{2\xi}{\mu^2}(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_{\zeta}})\tilde{\mathcal{I}}^{(\infty)}_2, \qquad (109)$$

where the components of the above KAM transformation $\tilde{\Psi}^{(\infty)}$ are evaluated at $x = y = I_1 = I_2 = 0$. To help in the understanding of this expression, we recall that, given a vector of basic frequencies $\Lambda = (\mu, \Omega_2)$, the values of $\zeta = (\xi, \eta)$ in (109) are related to Λ through the R-dependent vector-function $h = h^{(R)}$ introduced in lemma 5.2, i.e. $\zeta = h(\Lambda)$. Then, using the estimates provided by this lemma, the lower bounds $|\mu| = |\lambda_+| \geqslant (M^{(0)})^{\alpha/2}$ and $|\xi| = |h_1(\Lambda)| \geqslant (M^{(0)})^{\alpha/2}$, the explicit expression of $\breve{\mathcal{Z}}$ in (21), the bounds (59) on the partial derivatives of Z and those of (89), (90) and (91) on the components of $\tilde{\Psi}^{(\infty)}$, we easily obtain an estimate of the form

$$|q^{(\infty)} - \xi|_{\bar{\mathcal{E}}^{(\infty)}, \rho^{(\infty)}, 0} \leqslant c_6 (M^{(0)})^{1-9\alpha}. \qquad (110)$$

This motivates one to introduce the real set $\breve{\mathcal{E}}^{(\infty)} = \breve{\mathcal{E}}^{(\infty)}(R)$ defined as (see (101))

$$\breve{\mathcal{E}}^{(\infty)} := \{\Lambda \in \mathcal{E}^{(\infty)} : \xi = h_1(\Lambda) \geqslant (M^{(0)})^{\alpha/2}\}.$$

It is clear that if $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$ then we have a *real* analytic invariant torus of the initial system (10). Additionally, we introduce the R-dependent function $\Phi^{(\infty)}(\theta, \Lambda)$,

$$\Phi^{(\infty)} : \mathbb{T}^2 \times \breve{\mathcal{E}}^{(\infty)} \to \mathbb{T} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2, \qquad (111)$$

giving the parametrization (in the original phase space) of the 4D-Cantor invariant manifold of 2D-bifurcated elliptic tori. This parametrization is defined through the composition of the above changes (i)–(iv) evaluated at $x = y = I_1 = I_2 = 0$.

Remark 5.8. We recall that the curves in the Λ space defined by the condition $\xi = 0$, giving the (stable) periodic orbits of the family, do not change with the selected order of the normal form. This implies that the 'boundary' $\xi = 0$ of the set $\breve{\mathcal{E}}^{(\infty)}(R)$ does not change with R. See remark 4.2.

5.14. Whitney-smoothness with respect to Λ

After showing the persistence of a Cantor family of 2D-real bifurcated invariant tori of $\mathcal{H}$, labelled by $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, in this section we are going to prove the Whitney-C^∞ regularity with respect to Λ of this construction. More precisely, we show that the function $\Omega_1^{(\infty)}(\Lambda)$, giving the first component of the vector of intrinsic frequencies of these tori, and the vector-function $\Phi^{(\infty)}(\theta, \Lambda)$, giving their parametrization, admit a Whitney extension to functions C^∞ with respect to Λ and analytic with respect to θ. Albeit Whitney-smoothness is a very classical subject, in appendix A.3 we include a brief summary with the main definitions and results that we require.

In order to achieve these results, we apply the inverse approximation lemma A.9 to $\Omega_1^{(\infty)}$ as a limit of $\{\Omega_1^{(n)}\}_{n\geqslant 0}$ (see section 5.11) and to $\Phi^{(\infty)}$ as a limit of $\{\Phi^{(n)}\}_{n\geqslant 0}$ (see below). In what follows, we discuss the application of the lemma to $\Phi^{(\infty)}$ (this is the most involved case), but leave the details for $\Omega_1^{(\infty)}$ to the reader.

The sequence of (analytic) 'approximate' parametrizations $\{\Phi^{(n)}\}_{n\geqslant 0}$ is constructed in terms of the sequence of canonical transformation $\{\tilde{\Psi}^{(n)}\}_{n\geqslant 0}$ (see (88)) provided by the KAM iterative procedure. Thus, to define $\Phi^{(n)}$ we proceed analogously as we did for $\Phi^{(\infty)}$ in (111). Concretely, we have to compose $\tilde{\Psi}^{(n)}$, evaluated at $x = y = I_1 = I_2 = 0$, with the changes (i)–(iv) summarized in section 5.13.

We first consider the coordinate change (32) and, performing the same abuse of notation as in (109), we define, for each $n \geqslant 0$,

$$
\begin{aligned}
&\phi_j^{(n)}(\theta, \Lambda) := \theta_j + \tilde{\Theta}_j^{(n)} - \frac{2\xi}{\mu^2}(\partial^2_{J_j,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\left(\frac{\lambda_+}{2\xi}\tilde{\mathcal{X}}^{(n)} + \frac{1}{2}\tilde{\mathcal{Y}}^{(n)}\right), \qquad j = 1, 2,\\
&q^{(n)}(\theta, \Lambda) := \xi + \tilde{\mathcal{X}}^{(n)} - \frac{\xi}{\lambda_+}\tilde{\mathcal{Y}}^{(n)} - \frac{2\xi}{\mu^2}(\partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\tilde{\mathcal{I}}_1^{(n)} - \frac{2\xi}{\mu^2}(\partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\tilde{\mathcal{I}}_2^{(n)}, \qquad (112)\\
&J_1^{(n)}(\theta, \Lambda) := \mathbb{I}(\zeta) + \tilde{\mathcal{I}}_1^{(n)}, \qquad J_2^{(n)}(\theta, \Lambda) := 2\xi\eta + \tilde{\mathcal{I}}_2^{(n)}, \qquad p^{(n)}(\theta, \Lambda) := \frac{\lambda_+}{2\xi}\tilde{\mathcal{X}}^{(n)} + \frac{1}{2}\tilde{\mathcal{Y}}^{(n)},
\end{aligned}
$$

with all the components of $\tilde{\Psi}^{(n)}$ evaluated at $x = y = I_1 = I_2 = 0$ (see comments following (109) for a better understanding of these expressions). Moreover, for convenience, we also extend these definitions to the case $n = -1$ by setting $\tilde{\Psi}^{(-1)} := 0$. By using the bounds of section 5.10 on the transformations $\tilde{\Psi}^{(n)}$ (see also the comments linked to (110)) it is not difficult to obtain the following estimates,

$$
\begin{aligned}
|\phi_j^{(n)} - \phi_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} &\leqslant c_7(M^{(0)})^{19\alpha/2}2^{-n(2\tau+4)}(\bar{\kappa}^{(0)})^{2^n}, \qquad (113)\\
|q^{(n)} - q^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} &\leqslant c_7(M^{(0)})^{10\alpha}2^{-n(2\tau+4)}(\bar{\kappa}^{(0)})^{2^n},\\
|J_j^{(n)} - J_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} &\leqslant c_7(M^{(0)})^{23\alpha/2}2^{-n(2\tau+4)}(\bar{\kappa}^{(0)})^{2^n}, \qquad (114)\\
|p^{(n)} - p^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} &\leqslant c_7(M^{(0)})^{10\alpha}2^{-n(2\tau+4)}(\bar{\kappa}^{(0)})^{2^n},
\end{aligned}
$$

for $j = 1, 2$, $n \geqslant 0$ and R small enough, with $c_7 > 0$ independent of R.

After that, we apply change (19) to parametrizations (112). Thus, the coordinates ϕ_1 and J_1 remain unchanged and for the other ones we have, for $n \geqslant -1$,

$$
\begin{aligned}
&x_1^{(n)}(\theta, \Lambda) := \sqrt{2q^{(n)}}\cos\phi_2^{(n)}, \qquad y_1^{(n)}(\theta, \Lambda) := -\frac{J_2^{(n)}}{\sqrt{2q^{(n)}}}\sin\phi_2^{(n)} + p^{(n)}\sqrt{2q^{(n)}}\cos\phi_2^{(n)},\\
&x_2^{(n)}(\theta, \Lambda) := -\sqrt{2q^{(n)}}\sin\phi_2^{(n)}, \qquad y_2^{(n)}(\theta, \Lambda) := -\frac{J_2^{(n)}}{\sqrt{2q^{(n)}}}\cos\phi_2^{(n)} - p^{(n)}\sqrt{2q^{(n)}}\sin\phi_2^{(n)}.
\end{aligned}
$$

Then, using the above bounds on parametrizations (112) we obtain, for $j = 1, 2$ and $n \geqslant 0$,

$$|x_j^{(n)} - x_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} \leqslant c_8 (M^{(0)})^{19\alpha/2} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{115}$$

$$|y_j^{(n)} - y_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} \leqslant c_8 (M^{(0)})^{37\alpha/4} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{116}$$

for certain $c_8 > 0$ independent of R. Among the technical results on the weighted norm we have used here, we stress the mean value theorem of lemma A.2, combined with the lower bound $|q^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} \geqslant (M^{(0)})^{\alpha/2}/2$, which follows in a completely analogous way as done for $q^{(\infty)}$ in (109).

Finally, we apply the (partial) normal form transformation $\hat{\Psi} = \hat{\Psi}^{(R)}$ to the components of the parametrizations obtained after change (19) and we end up with the desired (R-dependent) sequence $\Phi^{(n)}$. In particular, we point out that

$$\Phi^{(-1)}(\theta, \Lambda) = \hat{\Psi}(\theta_1, \sqrt{2\xi}\cos\theta_2, -\sqrt{2\xi}\sin\theta_2, \mathbb{I}(\zeta), -\eta\sqrt{2\xi}\sin\theta_2, -\eta\sqrt{2\xi}\cos\theta_2),$$

where we recall that $\zeta = (\xi, \eta) = h(\Lambda)$. To bound $\Phi^{(n)} - \Phi^{(n-1)}$ we rely on the mean value theorem of lemma A.2. Hence, we use Cauchy estimates on the bounds of point (ii) of theorem 4.1 in order to control the size of the partial derivatives of $\hat{\Psi}$. According to the bounds of section 5.7 on the adapted system of coordinates, we observe that the Cauchy estimates on these partial derivatives can be done in such a way that the worst of them involves, at most, a denominator of order R^2—but not any power of $M^{(0)}(R)$ at all. Then, if we combine them with (113), (114), (115) and (116), we can easily establish the following bound,

$$|\Phi^{(n)} - \Phi^{(n-1)}|_{\bar{\mathcal{E}}^{(n)},\bar{\rho}_-^{(n+1)},0} \leqslant c_9 R^{-2} (M^{(0)})^{37\alpha/4} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{117}$$

for some $c_9 > 0$ independent of R.

Once we have bounded the 'convergence speed' of $\Phi^{(n)}$, we have to now control, in geometric form, the width of the complex widening of the set $\breve{\mathcal{E}}^{(\infty)}$ to which we can apply the nth step of the KAM process. Thus, we introduce the sequence of complex sets $\{W^{(n)}\}_{n\geqslant 0}$, $W^{(n)} \subset \mathbb{C}^2$, given by $W^{(n)} := \breve{\mathcal{E}}^{(\infty)} + r^{(n)}$, where the R-dependent quantities $r^{(n)} = r^{(n)}(R)$ are defined as

$$r^{(n)} := r^{(0)}\chi^n, \qquad r^{(0)} := \frac{(\bar{\delta}^{(0)})^{\tau+1}(M^{(0)})^{\alpha/2}}{2^{2\tau+4}} \left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{-\tau-1}, \qquad \chi := 2^{-3-2\tau}.$$

We notice that (95) implies that $\nu^{(n)} \geqslant r^{(n)}$, so that $W^{(n)} \subset \hat{\mathcal{E}}^{(n)} + \nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$.

At this point, we start verifying the conditions of the inverse approximation lemma A.9 for $\Phi^{(\infty)}$ as a limit of $\Phi^{(n)}$. We remark that besides the Λ-dependence, which is the only one taken into account for the Whitney part, the sequence $\Phi^{(n)}$ also depends in an analytic and periodic way on the variables $\theta \in \Delta_2(\rho^{(\infty)})$ (see (6) and (77)). Then, according to remark A.1, both the analytic and periodic dependence are preserved by the Whitney extension by simply dealing with θ as a parameter. Thus, we define $U^{(n)}(\theta, \Lambda) := \Phi^{(n-1)} - \Phi^{(-1)}$, for $n \geqslant 0$. By definition we have that $U^{(0)} = 0$ and, using (117),

$$|U^{(n)} - U^{(n-1)}|_{\Delta_2(\rho^{(\infty)})\times W^{(n-1)}} \leqslant c_9 R^{-2} (M^{(0)})^{37\alpha/4} 2^{-(n-1)(2\tau+4)} (\bar{\kappa}^{(0)})^{2^{n-1}}, \quad n \geqslant 0. \tag{118}$$

Our purpose is to show that, for any $\beta > 0$, there is $S = S(\beta) > 0$ (also dependent on R) so that (118) is bounded by $S(r^{(n-1)})^\beta$. Indeed, we have the following conditions on S:

$$S \geqslant 2^{2\tau+4+\beta} c_9 (\bar{\delta}^{(0)})^{-\beta(\tau+1)} R^{-2} (M^{(0)})^{37\alpha/4-\alpha\beta/2} \left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\beta(\tau+1)}$$
$$\times 2^{n(\beta-1)(2\tau+4)} (\bar{\kappa}^{(0)})^{2^{n-1}}, \qquad n \geqslant 1.$$

Due to the super-exponential term $(\bar{\kappa}^{(0)})^{2^{n-1}}$—compared with the geometric growth in $r^{(n-1)}$—it is easy to realize the existence of such S. Hence, lemma A.9 ensures that the limit vector-function $U^{(\infty)} = \Phi^{(\infty)} - \Phi^{(-1)}$ is of class Whitney-C^{β} with respect to $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, for any $\beta > 0$, and so is $\Phi^{(\infty)}$ (observe that $\Phi^{(-1)}$ is analytic in Λ). Consequently, using the Whitney extension theorem A.10, the function $\Phi^{(\infty)}$ can be extended to a C^{∞}-function of Λ in the whole $\mathbb{R}^2$. Abusing notation, we keep the name $\Phi^{(\infty)}$ for this extension. As pointed out before, it keeps the analytic and periodic dependence with respect to $\theta \in \Delta_2(\rho^{(\infty)})$.

Acknowledgments

The authors thank the anonymous referees and the Board Member for their valuable comments and remarks which have contributed to improving this paper. This work has been partially supported by the Spanish MCyT/FEDER grant MTM2006-00478.

Appendix A

In this final section we have compiled those contents that, in our opinion, are necessary for the self-containment of this paper, but which we have preferred not to include in the body of the paper in order to facilitate its readability. Concretely, in appendix A.1 we present the technical results on weighted norms we use to prove theorem 3.1. In appendix A.2 we prove a technical bound concerning the statement of theorem 4.1. Finally, in appendix A.3 we present a brief introduction to Whitney-smoothness.

A.1. Basic properties of the weighted norm

The following lemmas review some properties of the weighted norm $|\cdot|_{\rho,R}$ introduced in (3). These properties are completely analogous to those for the usual supremum norm.

In lemmas from A.1 to A.4 we discuss the bounds in terms of this weighted norm for the product of functions, partial derivatives (Cauchy estimates), composition of functions, the mean value theorem, estimates on Hamiltonian flows and on small divisors. In lemma A.5 we discuss the convergence of an infinite composition of canonical transformations. In lemma A.6 we give a technical result on the norm of the square root and, finally, in lemma A.7 we give another technical result referring to the norm $|\cdot|_R$ introduced at the end of section 2.

For most of these results we omit the proof, because it can be done simply by expanding the functions in Taylor–Fourier series (2) and then bounding the resulting expressions. For full details we refer to [43]. Throughout this section we use the notation introduced in section 2, sometimes without explicit mention.

Lemma A.1. *Let $f = f(\theta, x, I, y)$ and $g = g(\theta, x, I, y)$ be analytic functions defined in $\mathcal{D}_{r,s}(\rho, R)$ with 2π-periodic dependence in θ. Then we have the following.*

(*i*) $|f \cdot g|_{\rho,R} \leqslant |f|_{\rho,R} \cdot |g|_{\rho,R}$.

(*ii*) *For any* $0 < \delta \leqslant R$, $0 \leqslant \chi < 1$, $i = 1, \ldots, r$ *and* $j = 1, \ldots, 2r$ *we have*

$$|\partial_{\theta_i} f|_{\rho-\delta,R} \leqslant \frac{|f|_{\rho,R}}{\delta \exp(1)}, \qquad |\partial_{I_i} f|_{\rho,R\chi} \leqslant \frac{|f|_{\rho,R}}{(1-\chi^2)R^2}, \qquad |\partial_{z_j} f|_{\rho,R\chi} \leqslant \frac{|f|_{\rho,R}}{(1-\chi)R},$$

with $z = (x, y)$. All these bounds can be extended to the case in which f and g take values in $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$ (assuming that the matrix product of (i) is defined).

Lemma A.2. *Let us take $0 < \rho_0 < \rho$ and $0 < R_0 < R$ and consider analytic vector functions $\Theta^{(i)}$, $\mathcal{I}^{(i)}$, $\mathcal{X}^{(i)}$ and $\mathcal{Y}^{(i)}$ defined for $(\theta, x, I, y) \in \mathcal{D}_{r,m}(\rho_0, R_0)$, 2π-periodic in θ and taking values in $\mathbb{C}^{r'}$, $\mathbb{C}^{r'}$, $\mathbb{C}^{m'}$ and $\mathbb{C}^{m'}$, respectively, for $i = 0, 1$. We assume that $|\Theta^{(i)}|_{\rho_0,R_0} \leqslant \rho - \rho_0$, $|\mathcal{I}^{(i)}|_{\rho_0,R_0} \leqslant R^2$ and that $|\mathcal{Z}^{(i)}|_{\rho_0,R_0} \leqslant R$, for $i = 0, 1$, where $\mathcal{Z}^{(i)} = (\mathcal{X}^{(i)}, \mathcal{Y}^{(i)})$. Let $f(\theta', x', I', y')$ be a given analytic function defined in $\mathcal{D}_{r',m'}(\rho, R)$ and 2π-periodic in θ'. We introduce*

$$F^{(i)}(\theta, x, I, y) = f(\theta + \Theta^{(i)}, \mathcal{X}^{(i)}, \mathcal{I}^{(i)}, \mathcal{Y}^{(i)}), \qquad G(\theta, x, I, y) = F^{(1)} - F^{(0)}.$$

Then, we have

(i) $|F^{(i)}|_{\rho_0,R_0} \leqslant |f|_{\rho,R}, \quad i = 0, 1,$

(ii) $|G|_{\rho_0,R_0} \leqslant r'|\partial_\theta f|_{\rho,R}|\Theta^{(1)} - \Theta^{(0)}|_{\rho_0,R_0} + r'|\partial_I f|_{\rho,R}|\mathcal{I}^{(1)} - \mathcal{I}^{(0)}|_{\rho_0,R_0} + 2m'|\partial_z f|_{\rho,R}|\mathcal{Z}^{(1)} - \mathcal{Z}^{(0)}|_{\rho_0,R_0}$.

Lemma A.3. *Let $S = S(\theta, x, I, y)$ be a function such that ∇S is analytic in $\mathcal{D}_{r,m}(\rho, R)$ and 2π-periodic in θ. We also assume that*

$$|\nabla_\theta S|_{\rho,R} \leqslant R^2(1 - \chi^2), \qquad |\nabla_I S|_{\rho,R} \leqslant \delta, \qquad |\nabla_z S|_{\rho,R} \leqslant R(1 - \chi),$$

for certain $0 < \chi < 1$ and $0 < \delta < \rho$, with $z = (x, y)$. If we denote by Ψ_t^S the flow time t of the Hamiltonian system S, then it is defined as $\Psi_t^S : \mathcal{D}_{r,m}(\rho - \delta, R\chi) \to \mathcal{D}_{r,m}(\rho, R)$, for every $-1 \leqslant t \leqslant 1$. Moreover, if we write $\Psi_t^S - \mathrm{Id} = (\Theta_t^S, \mathcal{X}_t^S, \mathcal{I}_t^S, \mathcal{Y}_t^S)$ and $\mathcal{Z}_t^S = (\mathcal{X}_t^S, \mathcal{Y}_t^S)$, then all these components are 2π-periodic in θ and the following bounds hold for any $-1 \leqslant t \leqslant 1$,

$$|\Theta_t^S|_{\rho-\delta,R\chi} \leqslant |t||\nabla_I S|_{\rho,R}, \qquad |\mathcal{I}_t^S|_{\rho-\delta,R\chi} \leqslant |t||\nabla_\theta S|_{\rho,R}, \qquad |\mathcal{Z}_t^S|_{\rho-\delta,R\chi} \leqslant |t||\nabla_z S|_{\rho,R}.$$

Lemma A.4. *Let $f = f(\theta)$ be an analytic and 2π-periodic function in the r-dimensional complex strip $\Delta_r(\rho)$, for some $\rho > 0$, and $\{d_k\}_{k\in\mathbb{Z}^r\setminus\{0\}} \subset \mathbb{C}^*$ with $|d_k| \geqslant \gamma/|k|_1^\tau$, for some $\gamma > 0$ and $\tau > 0$. We expand f in the Fourier series, $f = \sum_{k\in\mathbb{Z}^r} f_k \exp(\mathrm{i}\langle k, \theta\rangle)$, and assume that the average of f is zero, i.e. $\langle f\rangle_\theta = f_0 = 0$. Then, for any $0 < \delta \leqslant \rho$, we have that the function g defined as*

$$g(\theta) = \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{f_k}{d_k} \exp(\mathrm{i}\langle k, \theta\rangle)$$

satisfies the bound

$$|g|_{\rho-\delta,0} \leqslant \left(\frac{\tau}{\delta \exp(1)}\right)^\tau \frac{|f|_{\rho,0}}{\gamma}.$$

Lemma A.5. *We consider strictly decreasing sequences of positive numbers $\rho^{(n)}$, $R^{(n)}$, a_n, b_n and c_n, defined for $n \geqslant 0$ and such that the series $A = \sum_{n\geqslant 0} a_n$, $B = \sum_{n\geqslant 0} b_n$ and $C = \sum_{n\geqslant 0} c_n$ are convergent. Additionally, for a given $0 < \delta \leqslant 1/2$, we define $\rho_-^{(n)} = \rho^{(n)} - \delta$, $R_-^{(n)} = R^{(n)} \exp(-\delta)$ and suppose that $\lim_{n\to+\infty} \rho_-^{(n)} = \rho^{(\infty)}$ and $\lim_{n\to+\infty} R_-^{(n)} = R^{(\infty)}$ are both positive and that*

$$a_n \leqslant \rho_-^{(n)} - \rho_-^{(n+1)}, \qquad b_n \leqslant (R_-^{(n)})^2 - (R_-^{(n+1)})^2, \qquad c_n \leqslant R_-^{(n)} - R_-^{(n+1)}. \tag{119}$$

Let $\Psi^{(n)} : \mathcal{D}_{r,s}(\rho^{(n+1)}, R^{(n+1)}) \to \mathcal{D}_{r,s}(\rho^{(n)}, R^{(n)})$ be a sequence of analytic canonical transformations with the following bounds for the components of $\Psi^{(n)} - \mathrm{Id}$:

$$|\Theta^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant a_n, \qquad |\mathcal{I}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant b_n, \qquad |\mathcal{Z}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant c_n.$$

If we define the composition $\tilde{\Psi}^{(n)} = \Psi^{(0)} \circ \cdots \circ \Psi^{(n)}$*, for any* $n \geqslant 0$*, then we have that* $\tilde{\Psi}^{(\infty)} = \lim_{n\to+\infty} \tilde{\Psi}^{(n)}$ *defines an analytic canonical transformation verifying the following.*

(i) $\tilde{\Psi}^{(\infty)} : \mathcal{D}_{r,s}(\rho^{(\infty)}, R^{(\infty)}) \to \mathcal{D}_{r,s}(\rho^{(0)}, R^{(0)})$.

(ii) *The components of* $\tilde{\Psi}^{(\infty)} - \mathrm{Id}$ *verify*

$$|\tilde{\Theta}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leqslant A, \qquad |\tilde{\mathcal{I}}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leqslant B, \qquad |\tilde{\mathcal{Z}}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leqslant C.$$

(iii) *If we define, for each* $n \geqslant 0$,

$$\Pi_n = \frac{1}{\delta}\left(\frac{ra_n}{\exp(1)} + \frac{rb_n}{(R^{(\infty)})^2} + \frac{4sc_n}{R^{(\infty)}}\right),$$

then the components of $\tilde{\Psi}^{(n)} - \mathrm{Id}$ *satisfy*

$$|\tilde{\Theta}^{(n)} - \tilde{\Theta}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leqslant a_n + A\Pi_n, \qquad |\tilde{\mathcal{I}}^{(n)} - \tilde{\mathcal{I}}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leqslant b_n + B\Pi_n,$$

$$|\tilde{\mathcal{Z}}^{(n)} - \tilde{\mathcal{Z}}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leqslant c_n + C\Pi_n.$$

Proof. In the proof we use the results on the weighted norm stated in lemmas A.1 and A.2. To prove the convergence of $\tilde{\Psi}^{(n)}$ we write

$$\tilde{\Psi}^{(n)} - \mathrm{Id} = \sum_{j=1}^{n} (\tilde{\Psi}^{(j)} - \tilde{\Psi}^{(j-1)}) + (\tilde{\Psi}^{(0)} - \mathrm{Id})$$

and study the absolute convergence of this sum, when $n \to +\infty$, by using the norm $|\cdot|_{\rho^{(\infty)},R^{(\infty)}}$. To do that, first we control the components of $\tilde{\Psi}^{(n)} - \mathrm{Id}$. We observe that

$$\begin{aligned}\tilde{\Psi}^{(n)} - \mathrm{Id} &= \Psi^{(n)} - \mathrm{Id} + (\tilde{\Psi}^{(n-1)} - \mathrm{Id}) \circ \Psi^{(n)} \\ &= \Psi^{(n)} - \mathrm{Id} + (\Psi^{(n-1)} - \mathrm{Id}) \circ \Psi^{(n)} + (\tilde{\Psi}^{(n-2)} - \mathrm{Id}) \circ \Psi^{(n-1)} \circ \Psi^{(n)}.\end{aligned}$$

Hence, reading this expression by components and proceeding by induction, we obtain the estimate

$$|\tilde{\Theta}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant \sum_{l=0}^{n} |\Theta^{(l)}|_{\rho^{(l+1)},R^{(l+1)}} \leqslant \sum_{l=0}^{n} a_l \leqslant A.$$

Similarly, we also have $|\tilde{\mathcal{I}}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant B$ and $|\tilde{\mathcal{Z}}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leqslant C$. At this point, if we assume an *a priori* convergence of $\tilde{\Psi}^{(n)}$, we clearly obtain the bounds in (ii) for $\tilde{\Psi}^{(\infty)}$. After that, we write

$$\tilde{\Psi}^{(j)} - \tilde{\Psi}^{(j-1)} = \tilde{\Psi}^{(j-1)} \circ \Psi^{(j)} - \tilde{\Psi}^{(j-1)} = \Psi^{(j)} - \mathrm{Id} + (\tilde{\Psi}^{(j-1)} - \mathrm{Id}) \circ \Psi^{(j)} - (\tilde{\Psi}^{(j-1)} - \mathrm{Id})$$

and consider this expression by components. For instance,

$$\tilde{\Theta}^{(j)} - \tilde{\Theta}^{(j-1)} = \Theta^{(j)} + \tilde{\Theta}^{(j-1)} \circ \Psi^{(j)} - \tilde{\Theta}^{(j-1)}.$$

Then, using the previous bound on $\tilde{\Psi}^{(j-1)}$, Cauchy estimates and the mean value theorem, we obtain

$$\begin{aligned}|\tilde{\Theta}^{(j)} - \tilde{\Theta}^{(j-1)}|_{\rho_-^{(j+1)},R_-^{(j+1)}} &\leqslant |\Theta^{(j)}|_{\rho^{(j+1)},R^{(j+1)}} + |\tilde{\Theta}^{(j-1)}|_{\rho^{(j)},R^{(j)}} \Bigg(r\frac{|\Theta^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{\exp(1)(\rho^{(j)} - \rho_-^{(j)})} \\ &\quad + r\frac{|\mathcal{I}^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{(R^{(j)})^2 - (R_-^{(j)})^2} + 2s\frac{|\mathcal{Z}^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{R^{(j)} - R_-^{(j)}} \Bigg) \\ &\leqslant a_j + A\left(\frac{ra_j}{\delta\exp(1)} + \frac{rb_j}{(R^{(j)})^2(1-\exp(-2\delta))} + \frac{2sc_j}{R^{(j)}(1-\exp(-\delta))}\right) \\ &\leqslant a_j + A\Pi_j,\end{aligned}$$

for any $j \geqslant 1$. To be more precise, we have bounded the partial derivatives of $\tilde{\Theta}^{(j-1)}$ in the domain $\mathcal{D}_{r,s}(\rho_-^{(j)}, R_-^{(j)})$ and then we have used hypothesis (119) to guarantee that $\Psi^{(j)}(\mathcal{D}_{r,s}(\rho_-^{(j+1)}, R_-^{(j+1)})) \subset \mathcal{D}_{r,s}(\rho_-^{(j)}, R_-^{(j)})$. Finally, we have also used that $R^{(j)} \geqslant R^{(\infty)}$ and that $(1-\exp(-x))^{-1} \leqslant 2/x$, whenever $0 < x \leqslant 1$. Analogously, we can derive bounds in (iii) for the remaining components. Finally, the convergence of $\sum_{j\geqslant 1} |\tilde{\Psi}^{(j)} - \tilde{\Psi}^{(j-1)}|_{\rho^{(\infty)}, R^{(\infty)}}$ follows immediately using these bounds. □

Lemma A.6. *Let $f(\theta, x, I, y)$ be an analytic function defined in $\mathcal{D}_{r,m}(\rho, R)$, 2π-periodic in θ and such that $|f|_{\rho,R} \leqslant L < 1$. Let $g(\theta, x, I, y)$ and $h(\theta, x, I, y)$ be given by*

$$g(\theta, x, I, y) = \sqrt{1 + f(\theta, x, I, y)}, \qquad h(\theta, x, I, y) = (\sqrt{1 + f(\theta, x, I, y)})^{-1}.$$

Then, one has $|g|_{\rho,R} \leqslant 2 - \sqrt{1-L}$ and $|h|_{\rho,R} \leqslant 1/\sqrt{1-L}$.

Proof. To prove both inequalities, we simply develop the square roots using the binomial expansion,

$$|g|_{\rho,R} \leqslant \sum_{j\geqslant 0} \left| \binom{1/2}{j} \right| L^j = 2 - \sqrt{1-L}, \qquad |h|_{\rho,R} \leqslant \sum_{j\geqslant 0} \left| \binom{-1/2}{j} \right| L^j = \frac{1}{\sqrt{1-L}}. \quad \square$$

The next relation between the norms is used to establish the bounds on $|Z|_{R^2}$ in (59) and in appendix A.2.

Lemma A.7. *Let $f(u, v)$ be an analytic function around the origin and $F(x, y)$ the same function written in terms of (x, y) through the changes $u = (x_1^2 + x_2^2)/2$ and $v = (y_1 x_2 - x_1 y_2)/2$, i.e. $F(x, y) = f(u, v)$. Then, for any $R > 0$ we have $|f|_{R^2} = |F|_R$.*

Proof. We consider the following expansion for $f(u, v)$,

$$f(u, v) = \sum_{\bar{k}\in\mathbb{Z}_+^2} a_{\bar{k}} 2^{|\bar{k}|_1} u^{\bar{k}_1} v^{\bar{k}_2},$$

for certain coefficients $a_{\bar{k}}$. By definition, $|f|_{R^2} = \sum_{\bar{k}} 2^{|\bar{k}|_1} |a_{\bar{k}}| R^{2|\bar{k}|_1}$. Then, we can write F as

$$F(x, y) = \sum_{k\in\mathbb{Z}_+^4} (-1)^{k_4} \binom{k_1 + k_2}{k_1} \binom{k_3 + k_4}{k_3} a_{(k_1+k_2, k_3+k_4)} x_1^{2k_1} x_2^{2k_2} (y_1 x_2)^{k_3} (y_2 x_1)^{k_4}.$$

We point out that all the monomials in the sum above are different for different k, so that

$$|F|_R = \sum_{k\in\mathbb{Z}_+^4} \binom{k_1 + k_2}{k_1} \binom{k_3 + k_4}{k_3} |a_{(k_1+k_2, k_3+k_4)}| R^{2|k|_1} = |f|_{R^2}. \quad \square$$

A.2. Bound on the term $\tilde{Z}^{(R)}$ of the normal form

As we pointed out at the end of section 4.1, the estimate $|\tilde{Z}^{(R)}|_{0,R} \leqslant \tilde{c} R^6$ in the statement of theorem 4.1 is not explicitly contained in [40]. In this section we show how this bound can be derived from the estimate $|\mathcal{Z}^{(R)}|_{0,R} \leqslant |\mathcal{H}|_{\rho_0,R_0}$ and the special structure of the normal form.

To establish this estimate on $\tilde{Z}^{(R)}$ we take advantage of the fact that the normal form can be expanded in powers of $(q, I, L/2)$, where $q = (x_1^2 + x_2^2)/2$ and $L = y_1 x_2 - x_1 y_2$. Concretely, $\tilde{Z}^{(R)}(x, I, y) = Z^{(R)}(q, I, L/2)$, with $Z^{(R)}(u)$ starting at degree three in $u = (u_1, u_2, u_3)$. We refer to point (iii) of theorem 4.1 for more details. Thus, our purpose is to now bound the norm $|Z^{(R)}|_{R^2}$, which corresponds to the weighted norm for the expansion of $Z^{(R)}(u)$ in powers of u. Then, using lemma A.7 we can relate the norms $|\tilde{Z}^{(R)}|_{0,R} = |Z^{(R)}|_{R^2}$.

Once we have fixed the value of $\varepsilon > 0$, we take any $0 < R \leqslant R^*$ and consider the following decomposition:

$$Z^{(R)} = \breve{Z} + \hat{Z}^{(R)},$$

where $\breve{Z}$ is independent of R and contains the affine terms in u_1 and u_3 of the normal form $Z^{(R)}$ as described in point (iii) of theorem 4.1 (see also the comments following the statement of the theorem). The term $\hat{Z}^{(R)}$ is a polynomial on u, but $\breve{Z}$ allows a general (analytic) power series expansion on u_2. But due to the fact that $\breve{Z}$ is independent of R, we easily have that there is $\breve{A}$ (independent of R) such that $|\breve{Z}|_{R^2} \leqslant \breve{A}R^6$ for any R small enough. For the remaining terms we only have $|\hat{Z}^{(R)}|_{R^2} \leqslant A$, with $A := |\mathcal{H}|_{\rho_0,R_0}$ also independent of R. In this case, we know that $\hat{Z}^{(R)}$ is a polynomial of degree less than or equal to $\lfloor r_{\text{opt}}(R)/2 \rfloor$, where $r_{\text{opt}}(R)$ depends on ε (see (16)). Let us assume that R is small enough such that this degree is bigger than three and expand:

$$\hat{Z}^{(R)} = \sum_{p=3}^{\lfloor r_{\text{opt}}(R)/2 \rfloor} \hat{Z}_p,$$

where $\hat{Z}_p = \hat{Z}_p(u)$ contains the terms of degree p in u of the normal form, except those included in $\breve{Z}$. We remark that the particular expression of the homogeneous polynomials $\hat{Z}_p$ is independent of R. By using Cauchy estimates we have the following bound for these terms:

$$|\hat{Z}_p|_{R^2} \leqslant A \frac{R^{2p}}{R_p^{2p}},$$

where $R_p = R_p(\varepsilon)$ denotes the first value of R for which $\lfloor r_{\text{opt}}(R)/2 \rfloor \geqslant p$. Concretely, we observe that

$$p = \left\lfloor 1 + \frac{1}{2} \exp\left(W \left(\log \left(\frac{1}{R_p^\alpha} \right) \right) \right) \right\rfloor,$$

where $\alpha = 1/(\tau + 1 + \varepsilon)$. By skipping the integer part, we obtain the bounds

$$\frac{1}{2} \exp\left(W \left(\log \left(\frac{1}{R_p^\alpha} \right) \right) \right) \leqslant p \leqslant 1 + \frac{1}{2} \exp\left(W \left(\log \left(\frac{1}{R_p^\alpha} \right) \right) \right).$$

Then, simple computations show that $(2(p-1))^{2(p-1)/\alpha} \leqslant R_p^{-1} \leqslant (2p)^{2p/\alpha}$. As a consequence, we obtain the following ε-dependent bound

$$|\hat{Z}_p|_{R^2} \leqslant A(2p)^{4p^2/\alpha} R^{2p}.$$

If we let $\varepsilon \to 0^+$, then we obtain a ε-independent bound. Then, we have

$$|\hat{Z}|_{R^2} \leqslant \sum_{p=3}^{\lfloor r_{\text{opt}}(R)/2 \rfloor} A(2p)^{4(\tau+1)p^2} R^{2p} = AR^6 \sum_{p=3}^{\lfloor r_{\text{opt}}(R)/2 \rfloor} (2p)^{4(\tau+1)p^2} R^{2p-6}.$$

In order to bound this last sum by an expression independent of R, we remark that, once we have fixed the value of R, then for all the indices p appearing in the above sum we have $R \leqslant R_p$. Consequently, by using the upper bound for R_p previously derived we obtain

$$|\hat{Z}|_{R^2} \leqslant AR^6 \sum_{p\geqslant 3} \frac{(2p)^{4(\tau+1)p^2}}{(2(p-1))^{2(p-1)(2p-6)(\tau+1+\varepsilon)}} \leqslant AR^6 \sum_{p\geqslant 3} \frac{e^{6(\tau+1)p}}{(2(p-1))^{4\varepsilon p^2+(12-16p)(\tau+1+\varepsilon)}} = \hat{A}R^6,$$

where we have used that $(p/(p-1))^p \leqslant e^{3/2}$. Therefore the convergence of $\hat{A} = \hat{A}(\varepsilon)$ is clear, for any $\varepsilon > 0$, and then we define $\tilde{c} = \tilde{c}(\varepsilon) := \breve{A} + \hat{A}$.

A.3. Whitney-smoothness

In this section we review the main definitions about Whitney-smoothness and the basic results on the topic we have used in section 5.14. See appendix 6 in [11] for a straightforward survey on the subject.

Definition A.8. *Let $A \subset \mathbb{R}^n$ be a closed set and $\beta > 0$ with $\beta \notin \mathbb{N}$. A Whitney-C^β function u on A—we shall write $u \in C^\beta_{Wh}(A)$—consists of a collection $u = \{u_q\}_{0 \leqslant |q|_1 \leqslant k}$, with $k = \lfloor \beta \rfloor$ and $q \in \mathbb{Z}^n_+$, of functions defined on A satisfying the following property: there exists $\hat{\gamma} > 0$ such that*

$$|u_q(x)| \leqslant \hat{\gamma}, \quad |u_q(x) - P_q(x, y)| \leqslant \hat{\gamma}|x - y|^{\beta - |q|_1}, \quad \forall x, y \in A, \ \forall q \in \mathbb{Z}^n_+, \ 0 \leqslant |q|_1 \leqslant k, \tag{120}$$

where $P_q(x, y)$ is analogous to the $(k - |q|_1)$th order Taylor polynomial of u_q. More precisely,

$$P_q(x, y) = \sum_{j=0}^{k-|q|_1} \sum_{|l|_1 = j} \frac{1}{l!} u_{q+l}(y)(x - y)^l, \qquad l \in \mathbb{Z}^n_+,$$

with the multi-index notation $l! = \prod_{i=1}^n l_i!$ and $(x-y)^l = \prod_{i=1}^n (x_i - y_i)^{l_i}$. The norm $\|u\|_{C^\beta_{Wh}(A)}$ is defined as the smallest $\hat{\gamma}$ for which (120) holds. If $u \in C^\beta_{Wh}(A)$ for all $\beta \notin \mathbb{N}$, we will refer to u as a Whitney-C^∞ function—we shall write $u \in C^\infty_{Wh}(A)$.

Of course, conditions (120) are not easy to fulfil for a given function defined on an arbitrary closed set A. However, in the case it is constructed as a limit of analytic functions, the next result provides a way to verify those properties (see [60] for a proof).

Lemma A.9 (Inverse approximation lemma). *Take a geometric sequence $r_j = r_0 \chi^j$, with $r_0 > 0$ and $0 < \chi < 1$. Let $A \subset \mathbb{R}^n$ be an open or closed set and define $W_j = A + r_j$, $j \in \mathbb{Z}_+$ (see (9)). Consider a sequence of real analytic functions $\{U^{(j)}\}_{j \in \mathbb{Z}_+}$, with $U^{(0)} = 0$, such that $U^{(j)}$ is defined in W_{j-1} and*

$$|U^{(j)} - U^{(j-1)}|_{W_{j-1}} \leqslant S r^\beta_{j-1}, \qquad j \in \mathbb{N},$$

for some constants $S \geqslant 0$ and $\beta > 0$, with $\beta \notin \mathbb{N}$. Then, there exists a unique function $U^{(\infty)}$, defined on A, which is of class Whitney-C^β and such that

$$\|U^{(\infty)}\|_{C^\beta_{Wh}(A)} \leqslant c_{\chi,\beta,n} S, \qquad \lim_{j \to +\infty} \|U^{(\infty)} - U^{(j)}\|_{C^\alpha_{Wh}(A)} = 0,$$

for all $\alpha < \beta$, where the constant $c_{\chi,\beta,n} > 0$ does not depend on A.

The following result states the classical *Whitney extension theorem*, claiming that Whitney-C^β functions defined on closed subsets of $\mathbb{R}^n$ can be extended to C^β functions on the whole space $\mathbb{R}^n$. See [56, 59].

Theorem A.10 (Whitney extension theorem). *For any $\beta > 0$, $\beta \notin \mathbb{N}$, and any closed set $A \subset \mathbb{R}^n$ there exists a (non-unique) linear extension operator $\mathcal{F}_\beta : C^\beta_{Wh}(A) \to C^\beta(\mathbb{R}^n)$, such that for each $u = \{u_q\}_q \in C^\beta_{Wh}(A)$ and $U = \mathcal{F}_\beta(u)$, we have, for all $0 \leqslant |q|_1 \leqslant \lfloor \beta \rfloor$,*

$$D^q U\big|_A = u_q, \qquad \|U\|_{C^\beta(\mathbb{R}^n)} \leqslant c_{\beta,n} \|u\|_{C^\beta_{Wh}(A)},$$

where $c_{\beta,n}$ does not depend on A. The norm in $C^\beta(\mathbb{R}^n)$ is the usual Hölder one, i.e. if $k = \lfloor \beta \rfloor$ then

$$\|U\|_{C^\beta(\mathbb{R}^n)} = \sup_{\substack{q \in \mathbb{Z}^n_+, \ |q|_1 \leqslant k \\ x \in \mathbb{R}^n}} \{|D^q U(x)|\} + \sup_{\substack{q \in \mathbb{Z}^n_+, \ |q|_1 = k \\ x, y \in \mathbb{R}^n, \ x \neq y}} \left\{ \frac{|D^q U(x) - D^q U(y)|}{|x - y|^{\beta - k}} \right\}.$$

If $\beta = +\infty$, then there is a (non-unique) linear extension operator $\mathcal{F} : C^\infty_{Wh}(A) \to C^\infty(\mathbb{R}^n)$, such that if $U = \mathcal{F}(u)$ then for all the derivatives—in the sense of Whitney—of u, $D^q U|_A = u_q$.

Remark A.1. It is worth remarking that the functions $U^{(j)}$, $j \in \mathbb{Z}_+$, and the limit function $U^{(\infty)}$ of lemma A.9 may depend in an analytic, smooth or periodic way on other variables. In such a case, these other variables might be thought of as parameters. Moreover, one can choose extension operators $\mathcal{F}_\beta$ and $\mathcal{F}$ preserving the analyticity (respectively, smoothness) as well as periodicity with respect to these parameters. See [11].

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