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DYNAMICS CLOSE TO A NON SEMI-SIMPLE 1:-1 RESONANT PERIODIC ORBIT

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Abstract. In this work, our target is to analyze the dynamics around the $1 : -1$ resonance which appears when a family of periodic orbits of a real analytic three-degree of freedom Hamiltonian system changes its stability from elliptic to a complex hyperbolic saddle passing through degenerate elliptic. Our analytical approach consists of computing, up to some given arbitrary order, the normal form around that resonant (or *critical*) periodic orbit.

Hence, dealing with the normal form itself and the differential equations related to it, we derive the generic existence of a two-parameter family of invariant 2D tori which bifurcate from the critical periodic orbit. Moreover, the coefficient of the normal form that determines the stability of the bifurcated tori is identified. This allows us to show the Hopf-like character of the unfolding: elliptic tori unfold "around" hyperbolic periodic orbits (case of *direct* bifurcation) while normal hyperbolic tori appear "around" elliptic periodic orbits (case of *inverse* bifurcation). Further, the parametrization of the main invariant objects as well as a global description of the dynamics of the normal form are also given.

1. **Introduction.** In this paper, the main topic is the study of the dynamics close to $1 : -1$ resonant periodic orbits of three-degree of freedom Hamiltonian systems. To be more precise: we consider a one-parameter family of periodic orbits of a real analytic three-degree of freedom Hamiltonian system, and assume that the orbits of the family are first linearly stable, for a *critical* value of the parameter the nontrivial (i. e., those different from one) characteristic multipliers of the corresponding periodic orbit collide on the unit circle (Krein collision, see appendix 29 of [2] and references therein) and then, if certain generic conditions are met, the characteristic multipliers leave out the unit circle to the complex plane. Hence the family looses its (linear) stability and the periodic orbits become *complex unstable*. In other words, the family changes from stable to complex-unstable by means of a passage through a *critical*, $1 : -1$ *resonant* or simply *resonant* periodic orbit.

These transitions can be found in several fields of science, from astronomy –in galactic dynamics [17], planetary theory [8]– to particle accelerators [10]; and not only in three degrees of freedom Hamiltonian systems, but also in higher dimensional problems (see [14]). This same mechanism of instabilization takes place in families of symplectic maps where a fixed point undergoes Krein collisions of its eigenvalues for some (critical) value of the parameter (see [3, 11, 19]).

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The outcome of this work can be summarized as follows: we rely on normal forms as the key tool of our approach, deriving up to *any* (arbitrary) order, a versal normal form of the Hamiltonian around the resonant periodic orbit. This involves the following steps: (i) we assume the Hamiltonian given in a suitable system of canonical coordinates which are adapted to the resonant periodic orbit; (ii) apply a canonical Floquet transformation to reduce the normal variational equations of the orbit to constant coefficients and (iii) proceed with the nonlinear reduction.

Hence, dealing with the normal form itself (i. e., we compute the normal form up to a given order and we skip the remainder) we show the generic unfolding of a two-parameter family of 2D-invariant tori (*Hamiltonian Hopf bifurcation*) and identify the coefficients which govern not only the bifurcation, but also its character: *direct* or *inverse*. In the case of direct bifurcation, there appear elliptic tori around complex-unstable periodic orbits, while in the case of inverse bifurcation, hyperbolic tori (but also parabolic and elliptic tori) unfold around stable periodic orbits. This study is completed with a description of the global dynamics of the normal form. We remark that this is not a merely qualitative (i. e., formal) process for, in addition, accurate parametrizations of the families of invariant tori and even of the invariant manifolds of the hyperbolic periodic orbits and hyperbolic tori are derived in this way. A numerical study of the above explained phenomenology has been recently done (for a direct bifurcation in the RTBP) in [15].

Indeed, as these invariant tori are obtained using an integrable approximation, the proof of their persistence (on Cantor sets) for the complete Hamiltonian involves the use of KAM techniques and is beyond the scope of the present paper. See the introduction of section 3 for more comments.

The contents of this paper are organized as follows. Section 2 tackles the computation of the normal form around the critical periodic orbit. The main result of this part, which is the normal form itself, is stated in theorem 2.1. Section 3 is devoted to the analysis of the dynamics of the normal form. In particular, theorem 3.1 establishes the unfolding of a two-parameter family of 2D invariant tori and proposition 3.1 states the normal stability of the bifurcated tori.

2. **Analytic approach.**

The purpose of this section is to describe briefly the main steps of the normal form process around the critical periodic orbit. A complete and constructive description of the process can be found in [16, 18]. In section 2.1 we give a precise formulation of the problem and state the "normalization theorem" in which the normal form is described (see theorem 2.1). In section 2.2 we introduce (local) adapted coordinates around the resonant periodic orbit. The purpose of this change is to separate the dynamics along the periodic orbit (described now by an angular variable and its conjugate action), from the movement in the "normal directions". Next, in section 2.3, a symplectic Floquet change is applied. The final goal is to arrive –through a symplectic 2π-periodic linear change–, to a "clean" Hamiltonian whose quadratic part is in Williamson's normal form with respect to its normal directions (see [1]). Later, the linearly reduced Hamiltonian is complexified (section 2.4) to simplify the structure of the homological equations arising in the nonlinear normalization process, which begins in section 2.5.

2.1. **Formulation of the problem.**

Let $H(\zeta)$ with $\zeta^* = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$, be a real three degree of freedom analytic Hamiltonian (asterisk will denote the transpose of a vector or a matrix) and consider its associated Hamiltonian system

$$\dot{\zeta} = J_3 \mathrm{grad} H(\zeta). \tag{1}$$

Henceforth, J_n will denote the matrix of the standard canonical n-form in $\mathbb{R}^{2n}$,

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

being I_n the $n \times n$ identity matrix.

Suppose that this system has a one-parameter family of nondegenerate periodic orbits, $\{\mathcal{M}_\sigma\}_{\sigma \in \mathbb{R}}$, such that for $\sigma = 0$ the corresponding orbit $\mathcal{M}_0$ (from now on, the *critical* or *resonant* periodic orbit) has an *irrational* (see definition 2.1) collision of its nontrivial Floquet (characteristic) multipliers. We recall that in Hamiltonian systems periodic orbits appear generically as one-parameter families parametrized by the energy (see [20]). For instance, we suppose that for $\sigma < 0$ the nontrivial multipliers of $\mathcal{M}_\sigma$ lie on the unit circle, they approach pairwise as σ goes to $\sigma = 0$, for this value they collide and separate towards the complex plane when $\sigma > 0$.

Definition 2.1. *Let $\lambda_0 \neq 1$ be a (double) nontrivial characteristic multiplier of the resonant periodic orbit $\mathcal{M}_0$ and let $\mu = 2\pi\kappa$ be its principal characteristic exponent (so $\lambda_0 = \mathrm{e}^{\mathrm{i}\mu}$). We say that the collision of characteristic multipliers on the unit circle is irrational if μ is not commensurable with 2π or, equivalently, if $\kappa \notin \mathbb{Q}$.*

Moreover, we assume *genericity* of the collision, in the following sense.

Definition 2.2. *Let $M(\mathcal{M}_0)$ be the monodromy matrix of the resonant periodic orbit $\mathcal{M}_0$. Hence,* $\mathrm{Spec}\,(M(\mathcal{M}_0)) = \{1, \lambda_0, 1/\lambda_0\}$. *The Krein collision will be called* generic *if the Jordan normal form of $M(\mathcal{M}_0)$ has the following block structure,*

$$J_{M(\mathcal{M}_0)} = \left(\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 1 & 1 & & & & \\ \hline & & \lambda_0 & 0 & & \\ & & 1 & \lambda_0 & & \\ \hline & & & & 1/\lambda_0 & 0 \\ & & & & 1 & 1/\lambda_0 \end{array}\right) \tag{2}$$

Remark 1. *Thus, one is assuming that none of the Jordan blocks of the monodromy matrix at the resonance is trivial (diagonal). In particular, the nontrivial character of the first block –corresponding to the eigenvalue equal to 1–, follows from the nondegeneracy of the family of periodic orbits. This is precisely the generic condition which allows to parametrize the family using the energy as a parameter.*

The main result of section 2 is the following normal form theorem.

Theorem 2.1. *Consider the three-degree of freedom Hamiltonian system (1). Let $\{\mathcal{M}_\sigma\}_{\sigma \in \mathbb{R}}$ be a one-parameter family of periodic orbits of this system with an irrational (so $\kappa \notin \mathbb{Q}$, see definition 2.1) and generic (see definition 2.2) Krein collision at $\sigma = 0$; also, let ω_1 denote the angular frequency of $\mathcal{M}_0$ and define $\omega_2 := \kappa\omega_1$.*

Then, given any $r \geq 3$, there exists a real analytic symplectic change: $(\xi, \eta) = \phi(\theta_1, x, I_1, y)$, defined in $\mathbb{S}^1 \times \mathcal{B}$ ($\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{B}$ a neighbourhood of the origin in $\mathbb{R}^5$) and taking values in a neighbourhood of $\mathcal{M}_0$, such that it casts the initial Hamiltonian into its normal form up to order r,

$$H \circ \phi(\theta_1, x, I_1, y) = Z^{(r)}(x, I_1, y) + R^{(r)}(\theta_1, x, I_1, y).$$

Here, $Z^{(r)}$ is the normal form up to order r and $R^{(r)}$ is the remainder (carrying higher order terms). The normal form is given by the sum

$$Z^{(r)} = \sum_{s=2}^{r} Z_s, \tag{3}$$

with

$$Z_2 = \omega_1 I_1 + \omega_2(y_1x_2 - y_2x_1) \pm \frac{1}{2}(y_1^2 + y_2^2),$$

where the sign $\pm$ in Z_2 is a characteristic of the collision. For $s \geq 3$, $Z_s = 0$ if s is odd and, when s is even, it is an homogeneous polynomial of degree $s/2$ in

$$\frac{1}{2}(x_1^2 + x_2^2), \quad I_1, \quad y_1x_2 - y_2x_1.$$

Although it is not pointed out in theorem 2.1, the change $(\xi, \eta) = \phi(\theta_1, x, I_1, y)$ yielding the normal form, depends on the order r.

We also remark that the detailed development of the reduction to the normal form given by (3) is lengthy and involved, due to the non semi-simple character of the nontrivial characteristic multipliers of the colision orbit. However, the method and main ideas are standard in normal form techniques. Of course, the full development is important if one is interested in the explicit (numerical) computation of the normal form (see [13] for the case of an elliptic periodic orbit) or in quantitative estimates (see introduction of section 3). As we focus our attention on the dynamics from the (truncated) normal form, we just recall the main steps and results and omit all the proofs. We refer the interested reader to [18] for all the details.

2.2. **Suitable coordinates around the critical periodic orbit.** As a first step, we shall introduce (local) adapted coordinates around the *critical* periodic orbit $\mathcal{M}_0$ through an analytic 2π-periodic in $\widetilde{\theta}$ change of variables,

$$\xi_i = \xi_i(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}), \qquad \eta_i = \eta_i(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}), \tag{4}$$

$i = 1, 2, 3$ and with $\widetilde{\xi}^* = (\widetilde{\xi}_1, \widetilde{\xi}_2)$, $\widetilde{\eta}^* = (\widetilde{\eta}_1, \widetilde{\eta}_2)$. Furthermore, we shall ask the change (4) to satisfy the following properties (see [5, 6] andreferences therein):

P1. It maps the product set $\mathbb{T}^1 \times \mathcal{O}$, where $\mathcal{O}$ is a five-dimensional open set around the origin, onto some (possibly small) neighbourhood $\mathcal{U}$ of $\mathcal{M}_0$. We shall denote by $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$, $n \in \mathbb{N}$, the standard n-torus. In particular $\mathbb{T}^1 \equiv \mathbb{S}^1$.

P2. The periodic orbit $\mathcal{M}_0$ is given by $\widetilde{\xi} = \widetilde{\eta} = 0$, $\widetilde{I} = 0$ and parametrized by $\widetilde{\theta}$. Moreover, if ω_1 is the angular frequency of $\mathcal{M}_0$, then the dynamics of $\widetilde{\theta}$ on the orbit is a linear periodic flow, $\dot{\widetilde{\theta}} = \omega_1$.

P3. The change (4) is symplectic with $\widetilde{\theta}$, $\widetilde{\xi}$ and $\widetilde{I}$, $\widetilde{\eta}$ the new conjugate positions and momenta respectively. So in these coordinates, the system (1) is transformed into another Hamiltonian system,

$$\dot{\widetilde{\theta}} = \frac{\partial \widetilde{H}}{\partial \widetilde{I}}, \qquad \dot{\widetilde{I}} = -\frac{\partial \widetilde{H}}{\partial \widetilde{\theta}}, \qquad \dot{\widetilde{\xi}}_i = \frac{\partial \widetilde{H}}{\partial \widetilde{\eta}_i}, \qquad \dot{\widetilde{\eta}}_i = -\frac{\partial \widetilde{H}}{\partial \widetilde{\xi}_i}, \qquad i = 1, 2. \tag{5}$$

For an example with an explicit construction of *local* canonical coordinates such like the ones just described, see [13].

The transformed Hamiltonian $\widetilde{H}$, defined in $\mathbb{T} \times \mathcal{O}$, is analytic and 2π-periodic in $\widetilde{\theta}$, so it can be expanded in a convergent Taylor series,

$$\widetilde{H}(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}) = \sum_{k,l,m} \widetilde{h}_{k,l,m}(\widetilde{\theta}) \widetilde{I}^k \widetilde{\xi}^l \widetilde{\eta}^m, \tag{6}$$

with $\widetilde{\xi}^* = (\widetilde{\xi}_1, \widetilde{\xi}_2)$, $\widetilde{\eta}^* = (\widetilde{\eta}_1, \widetilde{\eta}_2)$ and the standard multi-index notation $\widetilde{\xi}^l \widetilde{\eta}^m = \widetilde{\xi}_1^{l_1} \widetilde{\xi}_2^{l_2} \widetilde{\eta}_1^{m_1} \widetilde{\eta}_2^{m_2}$, which we shall use throughout the text. The index k, and the components of $l^* = (l_1, l_2)$, $m^* = (m_1, m_2)$ range over the nonnegative integers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, while the coefficients $\widetilde{h}_{k,l,m}(\widetilde{\theta})$ are analytic 2π-periodic functions in $\widetilde{\theta}$. If we restrict the system (5) to the periodic orbit $\mathcal{M}_0$, and take into account the expansion (6), we get

$$0 = \widetilde{h}'_{0,0,0,0,0}(\widetilde{\theta}), \quad \omega_1 = \widetilde{h}_{1,0,0,0,0}(\widetilde{\theta}), \quad 0 = \widetilde{h}_{0,e_i}(\widetilde{\theta}), \quad i = 1, 2, 3, 4,$$

(e_i is the i-th unit vector in $\mathbb{R}^4$), since by the condition *P2*, $\dot{\widetilde{\xi}} = \dot{\widetilde{\eta}} = 0$, $\dot{\widetilde{I}} = 0$ and $\dot{\widetilde{\theta}} = \omega_1$ on the periodic orbit $\mathcal{M}_0$. Then, from equations above, it follows that $\widetilde{h}_{0,0,0,0,0}(\widetilde{\theta}) \equiv$ const., so we can set $\widetilde{h}_{0,0,0,0,0} = 0$.

2.3. **Linear normalization.** The main result of this section states that, beyond the adapted coordinates, a new symplectic change (a "canonical Floquet" transformation) can be applied to reduce the *normal variational equations* around the critical orbit (that is, the linearized system in the normal directions) to a system with constant coefficients. Throughout the text, we shall refer as the "normal directions" those normal to the periodic orbit. Clearly, once an angle and its conjugate action have been introduced to describe the periodic orbit, the normal directions in the phase space will be the ones associated to the rest of the positions and their corresponding conjugate momenta.

LEMMA 2.1. *Assuming that the monodromy matrix of the resonant periodic orbit $\mathcal{M}_0$ has the Jordan block structure (2) (and hence genericity of the Krein collision, according to definition 2.2), the Hamiltonian $\widetilde{H}(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta})$, obtained after section 2.2, can be transformed by means of a symplectic change into*

$$\mathcal{H}(\theta_1, x, I_1, y) = \mathcal{H}_2(x, I_1, y) + \cdots, \tag{7}$$

here (θ_1, x, I_1, y) are the new symplectic coordinates and $\mathcal{H}_2$ is given by,

$$\mathcal{H}_2(x, I_1, y) = \omega_1 I_1 + \omega_2(y_1 x_2 - y_2 x_1) \pm \frac{1}{2}(y_1^2 + y_2^2) \tag{8}$$

where $\omega_2 = \kappa\omega_1$ (see theorem 2.1) and the sign $\pm$ in the above formula is a characteristic of the collision; in particular we also remark that $\mathcal{H}_2$ is free from angular dependence. The dots mean higher order terms in (x, I_1, y). Furthermore, the canonical transformation is linear in the normal directions $z^* = (x, y)$ and 2π-periodic in the angle $\theta_1 = \widetilde{\theta}$ (Floquet canonical reduction). In this sense, we shall say that the transformed Hamiltonian (7) is "linearly reduced" (with respect to the normal directions).

We note that the normal part of $\mathcal{H}_2$ (i. e., excluding the "rotor" $\omega_1 I_1$) agrees with the quadratic normal form in the classification given in [1] for the non-diagonalizable case. See [3, 18] for a proof.

REMARK 2. *After lemma 2.1 we should deal with two different cases which correspond to the plus or minus sign in (8). However, applying first the symplectic change,*

$$\begin{array}{lll} \theta_1 = \epsilon\theta_1', & x_1 = \epsilon x_1', & x_2 = x_2', \\ I_1 = \epsilon I_1', & y_1 = \epsilon y_1', & y_2 = y_2' \end{array}$$

to the transformed Hamiltonian $\mathcal{H}$ (see (7)) one may get rid of the $\pm$ in (8) if further the substitution $t \mapsto \epsilon t$ (which reverses the sign of the time when $\epsilon = -1$)

is allowed. In the forthcoming we shall assume that both transformations have been made so the $\pm$ sign will no longer appear (i. e., only the case $\epsilon = 1$ is considered). Moreover, the primes will be dropped and the names $\mathcal{H}$ for the linearly reduced Hamiltonian and $\mathcal{H}_2$ for its lower order terms are kept.

In this way, we arrive to a new, linearly reduced (in the sense stated in lemma 2.1) Hamiltonian, whose complete expansion can be written as,

$$\mathcal{H}(\theta_1, x, I_1, y) = \mathcal{H}_2(x, I_1, y) + \sum_{2l+|m|_1+|n|_1 \geq 3} \widehat{h}_{l,m,n}(\theta_1) I_1^l x^m y^n \tag{9}$$

($l \in \mathbb{Z}_+, m, n \in \mathbb{Z}_+^2$), where $\mathcal{H}_2$ is now (see remark 2 above) given by,

$$\mathcal{H}_2(x, I_1, y) = \omega_1 I_1 + \omega_2(y_1 x_2 - y_2 x_1) + \frac{1}{2}(y_1^2 + y_2^2). \tag{10}$$

2.4. **Complexification of the Hamiltonian.** Before going on with the nonlinear normalization, and in order to get the homological equations in a simpler form, it is convenient to introduce the following (complex) coordinates,

$$x_1 = \frac{q_1 - p_2}{\sqrt{2}} \quad x_2 = -\frac{q_1 + p_2}{\mathrm{i}\sqrt{2}}, \quad y_1 = \frac{q_2 + p_1}{\sqrt{2}}, \quad y_2 = -\frac{q_2 - p_1}{\mathrm{i}\sqrt{2}}. \tag{11}$$

These last relations define a linear canonical change which transforms the Hamiltonian (9) into

$$H(\theta_1, q, I_1, p) = H_2(q, I_1, p) + \sum_{2l+|m|_1+|n|_1 \geq 3} h_{l,m,n}(\theta_1) I_1^l q^m p^n, \tag{12}$$

where H_2 is (10) expressed in these coordinates (see (13)). As usual, we have put $q^* = (q_1, q_2)$, $p^* = (p_1, p_2)$ and $h_{l,m,n}(\theta_1)$ are analytic 2π-periodic functions.

Also, by direct substitution of (11) in the Hamiltonian (9), it can be seen that the quadratic part in (12) is,

$$H_2 = \omega_1 I_1 + \mathrm{i}\omega_2(q_1 p_1 + q_2 p_2) + q_2 p_1. \tag{13}$$

This will be the lowest-order term in our normal form. Note that, in the change (11) x and y will be real provided that $\bar{q}_1 = -p_2$ and $\bar{q}_2 = p_1$.

Remark 3. *If the above relations are assumed to hold and, as the complex Hamiltonian H is the transformed of a real Hamiltonian $\mathcal{H}$, it must be $\overline{H} = H$. More precisely, if we expand H in Poisson (Taylor-Fourier) series,*

$$H(\theta_1, q, I_1, p) = \sum_{k,l,m,n} h_{k,l,m,n} I_1^l \, q^m p^n \exp(\mathrm{i}k\theta_1), \tag{14}$$

with $k \in \mathbb{Z}$, then it is readily checked that the inverse change of (11) transforms H back to the Poisson series of a real function if and only if the relations:

$$\bar{h}_{k,l,m_1,m_2,n_1,n_2} = (-1)^{m_1+n_2} h_{-k,l,n_2,n_1,m_2,m_1}$$

hold between the coefficients of the expansion of H.

2.5. **Nonlinear normalization.** The Hamiltonian with the quadratic part given by (13) is suitable to start the normal form process. We notice that if this normalization is carried out up to any order, it leads to a generically divergent system (due to the small divisors involved). So, if we want a convergent Hamiltonian, we have to stop the normal form after a finite number of steps of the normalizing process.

Now, we sketch this reduction process. The most remarkable fact is that the non semi-simple character of the monodromy matrix of the critical orbit in the normal directions (the *generic* condition of definition 2.2), give rise to non semi-simple homological equations in the reduction process, which is not the standard context in normal forms. However, it is not strongly difficult to identify the removable and non-removable terms of the system. What is more complicate (than in the diagonal case) is to compute explicitly the normalizing transformation. Details of this reduction and a constructive algorithm to perform it are given in [16, 18].

A very natural way to compute the normalizing (canonical) transformation is to use the Lie series method to remove in an increasing way, and up to a given finite order, the "nonresonant" terms of the Hamiltonian. Thus, the generating function of the transformation is computed degree by degree by solving, at every step, an homological equation of the form:

$$\{G, H_2\} + Z = F, \tag{15}$$

where F contains the terms (of a given degree s) to be removed by a suitable G while Z stands for the non-removable terms (*the normal form*). Here, $\{\cdot,\cdot\}$ is the Poisson bracket. If we denote by $\mathfrak{E}$ the space of formal Poisson series defined as (14), the important point for us is to investigate the action of the linear (Lie) operator

$$\begin{array}{rcl} L_{H_2} : \mathfrak{E} & \longrightarrow & \mathfrak{E} \\ \chi & \mapsto & L_{H_2}\chi = \{\chi, H_2\}, \end{array}$$

and to give a complement of Range L_{H_2} in $\mathfrak{E}$. Given a monomial $g \in \mathfrak{E}$, of the form

$$g = I_1^l\, q^m\, p^n \exp(\mathrm{i}k\theta_1),$$

then

$$L_{H_2} g = \left(\Omega + m_1 \frac{q_2}{q_1} - n_2 \frac{p_1}{p_2}\right) g,$$

where Ω (*the small divisors*) is introduced as

$$\Omega \equiv \Omega_{k,|m|_1,|n|_1} = \mathrm{i}\omega_1 k + \mathrm{i}\omega_2(|m|_1 - |n|_1),$$

with $|m|_1 = |m_1| + |m_2|$ (similarly for $|n|_1$). Since the frequencies ω_1 and ω_2 are rationally independent, we have that $\Omega_{k,|m|_1,|n|_1} = 0$ if and only if $k = 0$ and $|m|_1 = |n|_1$. Then, a careful analysis of the structure of the homological equation (15) restricted to the monomials with $\Omega \neq 0$, shows that all of them can be removed by a suitable choice of G. This means that if there are non-removable monomials they have to satisfy $\Omega = 0$.

Thus, next step is to discuss the solvability of (15) for the terms for which the resonant condition $\Omega = 0$ is fulfilled. Those are given by

$$\mathcal{P} = \left\{ \sum_{l,M} \sum_{0 \le m_1, n_2 \le M} f_{l,m_1,M-m_1,M-n_2,n_2} I_1^l q_1^{m_1} q_2^{M-m_1} p_1^{M-n_2} p_2^{n_2} \right\}.$$

What we have to investigate is the action of L_{H_2} on $\mathcal{P}$, which we denote by $L : \mathcal{P} \to \mathcal{P}$. A standard way to compute a complement of Range L in $\mathcal{P}$ is to introduce an Hermitian product in $\mathcal{P}$. Thus, the following decomposition works,

$$\mathcal{P} = \operatorname{Range} L \oplus \operatorname{Ker} L^\dagger$$

where $L^\dagger$ is the adjoint operator of L. Given

$$F = \sum F_{l,m,n} I_1^l q^m p^n, \quad G = \sum G_{l',m',n'} I_1^{l'} q^{m'} p^{n'},$$

if we consider the Hermitian product (see [7]):

$$\langle F|G\rangle = \sum_{m_1,m_2,n_1,n_2} m_1!\, m_2!\, n_1!\, n_2!\, F_{l,m,m} \overline{G}_{l,m,n},$$

then $L^\dagger = \{\cdot, H_2^\dagger\}$, with $H_2^\dagger = q_1 p_2$. Next, introducing

$$\xi_1 = \mathrm{i}\,(q_1p_1 + q_2p_2)\,, \qquad \xi_2 = q_1 p_2,$$

we can check that

$$\operatorname{Ker} L^\dagger = \left\{ \sum_{l,M} \sum_{n=0}^{M} g_{l,n,M-n} I_1^l \xi_1^n \xi_2^{M-n} \right\}.$$

Hence, theorem 2.1 follows immediately since if we apply the inverse of (11), we get

$$\mathrm{i}(q_1p_1 + q_2p_2) = y_1x_2 - y_2x_1, \quad q_1p_2 = -\frac{1}{2}(x_1^2 + x_2^2),$$

and therefore, for $s \geq 3$ and even, Z_s in (3) depends on the real coordinates I_1, x, y in the form stated in the theorem. A careful analysis of the algebraic structure of the homological equations shows that the symmetries due to the complexification are preserved by the normal form process, and all the coefficients of $Z^{(r)}$ turn out to be real. Finally, we recall that due to the minus sign in the second of the formulas above, the coefficients of Z_s may differ in a sign when expressed in real or in complex coordinates. Also, we remark that an additional reversion in the time t, $t \mapsto -t$, is necessary if $\epsilon = -1$ after the linear reduction (see lemma 2.1 and remark 2).

3. **Dynamics of the normal form.** In this section, the normal form $Z^{(r)}$ is analyzed. Then, after the setting of the Hamiltonian equations corresponding to $Z^{(r)}$ and the discussion of their first integrals (section 3.1) we derive, in section 3.2, a parametrization of the family of periodic orbits and discuss their stability (lemma 3.1). Next, in section 3.3 we show that –under certain generic conditions which depend intrinsically on the low order terms of the normal form– there unfolds, "surrounding" the periodic orbits, a two-parameter family of two dimensional invariant tori. Furthermore, A study of the normal behaviour of such bifurcating tori is done, and the results are summarized in proposition 3.1. The global picture resembles the classical Andronov-Hopf bifurcation, in the sense that unfolded *stable* (elliptic) objects (2D-invariant tori in our case) appear around lower dimensional *unstable* ones –here, the periodic orbits of the family– (*direct case*), whereas conversely, *unstable* (hyperbolic) 2D-invariant tori may unfold around *stable* periodic orbits (*inverse case*). Nevertheless, in the current Hamiltonian case, elliptic and parabolic tori branch off as well (see proposition 3.1). Whether the former or the latter phenomenon takes place, depends again upon the nature of the low-order normal form. In the literature –see [21]–, this kind of bifurcation is known as the *Hamiltonian Andronov-Hopf bifurcation.* Next, in sections 3.4 and 3.5, we give parametrizations of the invariant manifolds of the hyperbolic periodic orbits of the

family and of the bifurcated (hyperbolic) 2D-invariant tori. Section 3.6 deals with the 3D-invariant tori branching off the 2D-elliptic tori of proposition 3.1.

However, it is worth realizing that if only a qualitative description is needed, all these forementioned objects, and the dynamics generated from them, can be detected using the normal form (16) up to an order four (hence $r = 2$; see [9]). Nevertheless, if one looks for accurate parametrizations of those backbone dynamical invariants, then they must be computed from a normal form of higher order and therefore, our approach describes the (local) dynamics around the non semi-simple resonant periodic orbit not only qualitatively, but quantitatively as well, in the sense that allows all these computations effectively and up to any arbitrary order.

The questions related with the effect that the nonintegrable remainder has on the dynamics of the normal form are not treated here. A very natural one is the derivation, as a function of the distance to the critical orbit, of the "optimal" order of the normal form such that it minimizes the size of the remainder in a given neighbourhood of the resonant periodic orbit. This problem is tackled in [16, 18], and we only mention that classical exponentially small estimates for the size of the remainder of the normal form are not obtained in this case.

The most important question is the persistence of the bifurcated invariant tori detected from the normal form. To do that, one cannot avoid getting involved with KAM methods (see [4] and references therein for a wide description of these techniques). What is natural to expect is that these tori persist on Cantor sets for the parameters, with estimates on the size of the (relative) measure of the destroyed tori controlled by the size of the nonintegrable remainder (compare with the case of exponentially small estimates for the remainder of the normal form around an elliptic torus in [12]). A proof of this fact requires nondegeneracy conditions on the basic frequencies (see remark 5), and a careful control of the normal frequency of the 2D-elliptic tori. However, we want to stress that, as we have to deal with invariant objects that are very close to a degenerate (non semi-simple) orbit, this fact introduces additional difficulties forcing to modify the standard KAM iterative methods to achieve the desired estimates for the measure. For a proof in the case of a direct bifurcation see [18]. We hope to give a complete treatment of the problem (covering also the inverse case) in a future work.

3.1. **Hamiltonian equations of the truncated normal form.** From now on we shall concentrate on the study of the normal form $Z^{(r)}$, skipping the remainder $R^{(r)}$ off and working only with *real* coordinates throughout. Hence, in view of theorem 2.1, $Z^{(r)}$ can be put into the form:

$$Z^{(r)}(x, I_1, y) = \omega_1 I_1 + \omega_2 y \times x + \frac{1}{2}|y|_2^2 + \mathcal{Z}_r\left(\frac{1}{2}|x|_2^2, I_1, y \times x\right), \tag{16}$$

with the notation,

$$|x|_2 = (x_1^2 + x_2^2)^{1/2}, \quad |y|_2 = (y_1^2 + y_2^2)^{1/2}, \quad y \times x = x_2 y_1 - x_1 y_2$$

and $\mathcal{Z}_r(u, v, w)$ being a polynomial of degree $\lfloor r/2 \rfloor$ (we use $\lfloor x \rfloor$ to denote the integer part of $x \in \mathbb{R}$), beginning with quadratic terms. We shall express it as

$$\mathcal{Z}_r(u, v, w) = \frac{1}{2}(au^2 + bv^2 + cw^2) + duv + euw + fvw + \mathcal{F}_r(u, v, w) \tag{17}$$

with

$$\mathcal{F}_r(u, v, w) = \sum_{3 \le j+m+n \le \lfloor r/2 \rfloor} f_{j,m,n} u^j v^m w^n, \tag{18}$$

if $r \geq 6$ or zero otherwise.

Actually, the coefficients of the term of degree two are those which will play an essential rôle in the dynamics of $Z^{(r)}$. It becomes clear throughout the main results of this section: lemma 3.1, theorem 3.1 and proposition 3.1.

Now, if we define

$$\eta(x, I_1, y) := \left(\frac{1}{2}|x|_2^2, I_1, y \times x\right),$$

the corresponding Hamiltonian equations can be written in the form

$$\begin{array}{rcl} \dot{\theta}_1 & = & \omega_1 + \partial_2 \mathcal{Z}_r \circ \eta, \\ \dot{I}_1 & = & 0, \\ \dot{x}_1 & = & \omega_2 x_2 + y_1 + x_2 \partial_3 \mathcal{Z}_r \circ \eta, \\ \dot{x}_2 & = & -\omega_2 x_1 + y_2 - x_1 \partial_3 \mathcal{Z}_r \circ \eta, \\ \dot{y}_1 & = & \omega_2 y_2 - x_1 \partial_1 \mathcal{Z}_r \circ \eta + y_2 \partial_3 \mathcal{Z}_r \circ \eta, \\ \dot{y}_2 & = & -\omega_2 y_1 - x_2 \partial_1 \mathcal{Z}_r \circ \eta - y_1 \partial_3 \mathcal{Z}_r \circ \eta. \end{array} \tag{19}$$

where $\partial_i \mathcal{Z}_r$ is the partial derivative of the function $\mathcal{Z}_r$ with respect the i-th variable. The system above is integrable, since it can be shown that the three functions

$$\mathcal{I}_1 = I_1, \quad \mathcal{I}_2 = y \times x \quad \text{and} \quad \mathcal{I}_3 = \frac{1}{2}|y|_2^2 + \mathcal{Z}_r\left(\frac{1}{2}|x|_2^2, I_1, y \times x\right) \tag{20}$$

are, outside the zero measure set defined by $\{y_1 = 0,\, y_2 = 0,\, \partial_1 \mathcal{Z}_r = 0\}$, three functionally independent integrals in involution of the system (19).

3.2. **Parametrization of the family of periodic orbits.** These Hamiltonian equations have a one-parameter family of periodic orbits given by

$$\mathcal{M}_\sigma : \left\{ \begin{array}{l} \theta_1 = (\omega_1 + \partial_2 \mathcal{Z}_r(0, I_1, 0))t + \theta_1^0, \\ I_1 = \sigma, \\ x_1 = x_2 = y_1 = y_2 = 0. \end{array} \right. \tag{21}$$

This implies that the action I_1 is a good parameter for the (local) description of the initial family of periodic orbits. So we can identify $\sigma \equiv I_1$ as the parameter and, in the forthcoming, denote the family by $\{\mathcal{M}_{I_1}\}_{I_1 \in \mathbb{R}}$.

REMARK 4. *With the parametrization (21), the "twist" condition (see* [20]*)* $\omega'(0) \neq 0$*, requiring the angular frequency to move with the parameter of the family, can be expressed as* $\partial_{2,2} \mathcal{Z}_r(0, 0, 0) = b \neq 0$.

It turns out that, if the coefficient d in (17) is different from zero (this is a generic condition that will be assumed in the sequel), the stability of the family $\{\mathcal{M}_{I_1}\}_{I_1 \in \mathbb{R}}$ depends, for $|I_1|$ small, on the product dI_1. This is stated in the next lemma.

LEMMA 3.1. *If the coefficient d of the polynomial $\mathcal{Z}_r$ given by (17) is $d \neq 0$, then for $|I_1|$ small enough, the periodic orbits in $\{\mathcal{M}_{I_1}\}_{I_1 \in \mathbb{R}}$ are complex-unstable when $dI_1 < 0$ or (linearly) stable when $dI_1 > 0$.*

Proof. To compute the characteristic exponents of the periodic orbits, we write down the variational equations of (19) around $\mathcal{M}_{I_1}$. Using the parametrization (21), one may check that in the normal directions (x, y) these equations are given by the linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 & 1 & 0 \\ -\sigma_2 & 0 & 0 & 1 \\ -\sigma_1 & 0 & 0 & \sigma_2 \\ 0 & -\sigma_1 & -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix},$$

with σ_1, σ_2 defined by,

$$\begin{aligned} \sigma_1 &:= \partial_1 \mathcal{Z}_r(0, I_1, 0) = dI_1 + \mathrm{O}(I_1^2), \\ \sigma_2 &:= \omega_2 + \partial_3 \mathcal{Z}_r(0, I_1, 0) = \omega_2 + fI_1 + \mathrm{O}(I_1^2) \end{aligned} \tag{22}$$

and then the characteristic exponents of the periodic orbits are,

$$\begin{aligned} \alpha_{I_1}^{\pm} &= \mathrm{i}\sigma_2 \pm \sqrt{-\sigma_1} \\ &= \mathrm{i}(\omega_2 + fI_1 + \mathrm{O}(I_1^2)) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)}, \\ \beta_{I_1}^{\pm} &= -\mathrm{i}\sigma_2 \pm \sqrt{-\sigma_1} \\ &= -\mathrm{i}(\omega_2 + fI_1 + \mathrm{O}(I_1^2)) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)}. \end{aligned}$$

Thus, if $|I_1|$ is sufficiently small, the sign of the terms inside the square roots at the expansions for $\alpha_{I_1}^{\pm}$ and $\beta_{I_1}^{\pm}$ in the above formulas, depends on the sign of dI_1 in the way described by the lemma. □

3.3. **An unfolding of a two-parameter family of 2D-invariant tori.** Generically, this collision of characteristic multipliers carries on quasiperiodic bifurcation phenomena. These may be described using the normal form $Z^{(r)}$.

Before, to simplify the identification of the requested solutions, it is convenient to introduce new coordinates through the change:

$$\begin{aligned} x_1 &= \sqrt{2q}\cos\theta_2, & y_1 &= -\frac{I_2}{\sqrt{2q}}\sin\theta_2 + p\sqrt{2q}\cos\theta_2, \\ x_2 &= -\sqrt{2q}\sin\theta_2, & y_2 &= -\frac{I_2}{\sqrt{2q}}\cos\theta_2 - p\sqrt{2q}\sin\theta_2, \end{aligned} \tag{23}$$

with $q > 0$. Transformation (23) is canonical, for one immediately verifies: $\mathrm{d}\theta_1 \wedge \mathrm{d}I_1 + \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}q \wedge \mathrm{d}p$ and is properly defined and regular except in the set $x_1 = x_2 = 0$. It introduces a second action I_2, together with its conjugate angle θ_2 while q and p are the new normal position and its conjugate momentum respectively; in these coordinates, the Hamiltonian $Z^{(r)}$ takes the form,

$$Z^{(r)}(\theta_1, \theta_2, q, I_1, I_2, p) = \omega_1 I_1 + \omega_2 I_2 + qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2). \tag{24}$$

Assuming, as in lemma 3.1, $d \neq 0$, the Hamiltonian system corresponding to (24)

$$\begin{aligned} \dot\theta_1 &= \omega_1 + \partial_2 \mathcal{Z}_r(q, I_1, I_2), \\ \dot\theta_2 &= \omega_2 + \frac{I_2}{2q} + \partial_3 \mathcal{Z}_r(q, I_1, I_2), \\ \dot q &= 2qp, \\ \dot I_1 &= 0, \\ \dot I_2 &= 0, \\ \dot p &= -p^2 + \frac{I_2^2}{4q^2} - \partial_1 \mathcal{Z}_r(q, I_1, I_2), \end{aligned} \tag{25}$$

has a two-parameter family of bifurcated 2D-invariant tori. Theorem 3.1 sets a precise formulation of this assertion.

THEOREM 3.1. *If the coefficient of d of $\mathcal{Z}_r$ (see equation (17)) is $d \neq 0$ there exists a real analytic function $\mathcal{I} : \mathfrak{D} \subset \mathbb{R}^2 \to \mathbb{R}$, $\mathfrak{D}$ a neighbourhood of $(0,0)$, defined implicitly by the equation*

$$\eta^2 = \partial_1 \mathcal{Z}_r\left(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta\right),$$

with $\mathcal{I}(0,0)=0$ *and such that, for* $(\xi,\eta)\in\mathfrak{D}$*, the two-dimensional torus*

$$\mathcal{T}_{\xi,\eta}=\{(\theta,q,I,p)\in\mathbb{T}^2\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}: q=\xi,\ I_1=\mathcal{I}(\xi,\eta),\ I_2=2\xi\eta,\ p=0\} \quad (26)$$

is invariant under the flow of (25) with parallel dynamics determined by the vector $\Omega^*=(\Omega_1,\Omega_2)$ *of intrinsic frequencies:*

$$\begin{array}{rcl}\Omega_1(\xi,\eta) & = & \omega_1+\partial_2\mathcal{Z}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta),\\ \Omega_2(\xi,\eta) & = & \omega_2+\eta+\partial_3\mathcal{Z}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta);\end{array} \quad (27)$$

moreover, the corresponding invariant tori of (16) are real whenever $\xi>0$*.*

Proof. It follows directly by substitution in equations (25), whereas the last point about the real character of the invariant tori follows from (26) and the change (23). Here we only stress that the condition $d\neq 0$ (the non-degeneracy of the transition) is the necessary hypothesis for the implicit analytic function $\mathcal{I}$ to exist in a neighbourhood of $(0,0)$, since $\partial^2_{1,2}\mathcal{Z}_r(0,0,0)=d$. Finally, transformation (23) shows the real character of the (corresponding) tori of (16) for $\xi>0$. □

Then $\{\mathcal{T}_{\xi,\eta}\}_{(\xi,\eta)\in\mathfrak{D}}$ with $\xi>0$ constitutes a two-parameter family of real invariant tori filled up with quasiperiodic solutions of the system (25). Whence, changing back to rectangular (with respect to the normal directions) coordinates by means of (23), one obtains a family of two-parameter quasiperiodic solutions winding 2D-real invariant tori of (16). Explicitly:

$$\begin{array}{rclrcl}\theta_1 & = & \Omega_1(\xi,\eta)t+\theta_1^0, & I_1 & = & \mathcal{I}(\xi,\eta),\\ x_1 & = & \sqrt{2\xi}\cos(\Omega_2(\xi,\eta)t+\theta_2^0), & y_1 & = & -\sqrt{2\xi}\,\eta\sin(\Omega_2(\xi,\eta)t+\theta_2^0),\\ x_2 & = & -\sqrt{2\xi}\sin(\Omega_2(\xi,\eta)+\theta_2^0), & y_2 & = & -\sqrt{2\xi}\,\eta\cos(\Omega_2(\xi,\eta)t+\theta_2^0).\end{array}$$

Using the expressions for $\mathcal{Z}_r$ given by (17) and (18), a formal expansion of the implicit function $\mathcal{I}$ can easily be derived. Up to second order in ξ, η one gets:

$$\mathcal{I}(\xi,\eta)=-\frac{a}{d}\xi-\frac{1}{d}\left(3f_{3,0,0}-\frac{2af_{2,1,0}}{d}+\frac{a^2f_{1,2,0}}{d^2}\right)\xi^2-\frac{2e}{d}\xi\eta+\frac{1}{d}\eta^2+\mathrm{O}_3(\xi,\eta) \quad (28)$$

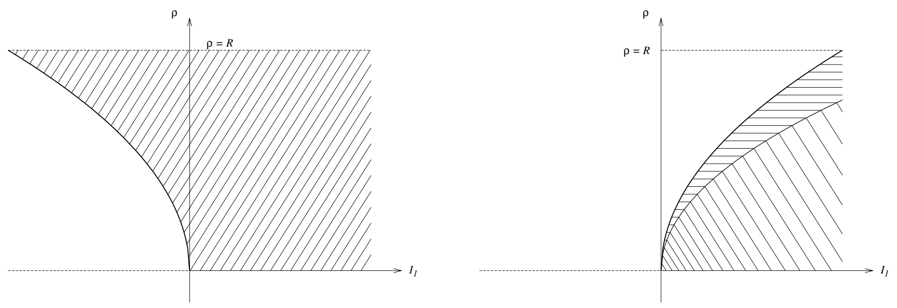
and then substitution in (27) yields, for the frequencies Ω_1, Ω_2,

$$\begin{array}{rcl}\Omega_1(\xi,\eta) & = & \omega_1+\left(d-\frac{ab}{d}\right)\xi\\ & + & \left(-\frac{3b}{d}f_{3,0,0}-\frac{a^2b}{d^3}f_{1,2,0}+\frac{2ab}{d^2}f_{2,1,0}+f_{2,1,0}-\frac{2a}{d}f_{1,2,0}+\frac{3a^2}{d^2}f_{0,3,0}\right)\xi^2\\ & + & \left(-\frac{2eb}{d}+2f\right)\xi\eta+\frac{b}{d}\eta^2+\mathrm{O}_3(\xi,\eta),\end{array} \quad (29)$$

$$\begin{array}{rcl}\Omega_2(\xi,\eta) & = & \omega_2+\left(e-\frac{af}{d}\right)\xi+\eta\\ & + & \left(-\frac{3f}{d}f_{3,0,0}-\frac{a^2f}{d^3}f_{1,2,0}+\frac{2af}{d^2}f_{2,1,0}+f_{2,0,1}-\frac{a}{d}f_{1,1,1}+\frac{a^2}{d^2}f_{0,2,1}\right)\xi^2\\ & + & \left(2c-\frac{2ef}{d}\right)\xi\eta+\frac{f}{d}\eta^2+\mathrm{O}_3(\xi,\eta).\end{array} \quad (30)$$

Similarly as in the lemma 3.1 for the stability of periodic orbits, the normal character (elliptic, hyperbolic) of the unfolded tori has to do with the sign of one of the coefficients of the polynomial $\mathcal{Z}_r$ (see proposition below).

Remark 5. *In view of the expansions (29) and (30) one easily computes* $\det \mathrm{D}_\zeta\Omega=d-ab/d+\mathrm{O}_1(\xi,\eta)$ *–with* $\zeta^*=(\xi,\eta)$*,* $\Omega^*=(\Omega_1,\Omega_2)$*–, so the family of invariant tori will be nondegenerated (in the Kolmogorov's sense) under the condition* $d^2\neq ab$*.*



(a) $a > 0$ (direct bifurcation) (b) $a < 0$ (inverse bifurcation)

FIGURE 1. Qualitative sketch of the size of the 2D bifurcated tori and their stability in the case $d > 0$. See the text for details.

PROPOSITION 3.1. *With the assumptions of theorem 3.1 –including the reality condition $\xi > 0$– and if the coefficient a of the polynomial $\mathcal{Z}_r$ in (17) is $a \neq 0$; the type of the bifurcation is determined by the sign of the coefficient a.*

Case 1. If $a > 0$; besides the elliptic tori around stable periodic orbits –which correspond to excitations in their normal elliptic directions– there appear elliptic tori around complex-unstable periodic orbits.

Case 2. If $a < 0$, then, hyperbolic invariant tori unfold around stable periodic orbits. In this case, the family described in theorem 3.1 contains also elliptic tori (of the same nature than those in the previous case) and parabolic tori.

By analogy with the classical Andronov-Hopf bifurcation, the former and the latter cases in the proposition are often referred as the "direct" and the "inverse" bifurcation respectively.

Proof (of proposition 3.1). Consider the system (25) and the family of invariant tori $\{\mathcal{T}_{\xi,\eta}\}_{(\xi,\eta)}$ of theorem 3.1. Around one of these tori, the first variational equations in the normal directions are given by the linear system,

$$\begin{array}{rcl} \dot{X} & = & 2\xi Y, \\ \dot{Y} & = & -\left[2\frac{\eta^2}{\xi} + a + \partial^2_{1,1}\mathcal{F}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta)\right] X \end{array}$$

whose eigenvalues (the characteristic exponents of the torus) are:

$$\mu_{\pm}(\xi,\eta) = \pm\left[-4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1}\mathcal{F}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta)\right]^{1/2}. \tag{31}$$

As $\xi\partial^2_{1,1}\mathcal{F}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta)$ is $\xi\mathrm{O}_1(\xi,\eta)$, at least –in a small neighbourhood of $(\xi,\eta) = (0,0)$–, the normal behaviour of the tori is determined by the sign of the first two terms inside the square root, i. e. by: $-4\eta^2 - 2a\xi$. In particular, if the coefficient a is positive (case 1) then the family only holds the elliptic invariant tori, whilst for negative values of a (case 2), elliptic and hyperbolic tori will be present, but parabolic tori will appear as well. Indeed, if one considers the equation: $2\eta^2 + a\xi + \xi\partial^2_{1,1}\mathcal{F}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) = 0$, just the implicit function theorem applied at $(\xi,\eta) = (0,0)$ shows the existence –in the space of parameters (ξ,η)–, of a path $\xi = g(\eta)$ giving rise to a one-parameter family of parabolic tori. Of course, the same can be done when $a < 0$ but then ξ, as a function of η, will take locally (i. e., in a small neighbourhood of the origin) only negative values, against the condition for real invariant tori.

To discuss the position –relative to the family of periodic orbits– of the bifurcated invariant tori, one looks at (28) and realizes that for (ξ, η) given, the action I_1 of the corresponding invariant torus can be expressed as

$$I_1 = -\frac{a}{d}\xi + \frac{1}{d}\eta^2 + \xi \mathrm{O}_1(\xi, \eta) + \mathrm{O}_3(\xi, \eta);$$

hence, the sign of dI_1 is determined locally by the first two terms. In particular, for $a > 0$ (case 1) dI_1 can take positive or negative values so elliptic bifurcated tori of the first case unfold "around" both stable and unstable periodic orbits. On the contrary, for $a < 0$ the sign of dI_1 is (locally) always positive and therefore, in the second case, bifurcated hyperbolic, elliptic and parabolic tori appear around –in the sense just stated– stable periodic orbits of the family. This ends the proof. □

In figure 1 the bifurcation pattern is sketched in both contexts for $d > 0$ (the figures for $d < 0$ follow straightforward). In these plots complex-unstable periodic orbits lie on the negative horizontal semiaxis (in dashed lines) and the stable ones in the positive horizontal semiaxis. In figure (a) the shaded area corresponds to the domain of existence of elliptic invariant tori. In figure (b) the regions shaded obliquely and horizontally are the domains of the elliptic and hyperbolic tori respectively whilst the separating curve holds parabolic tori. Here ρ, $0 < \rho < R$, is defined by $\rho := (x_1^2 + x_2^2)^{1/2}$ (so $\rho = (2\xi)^{1/2}$, according to (23)). It can be thought of as the radius of the invariant torus $\mathcal{T}_{\xi,\eta}$ in the normal directions x_1, x_2. R is the "maximum allowed radius" and is determined by the domain $\mathfrak{D}$ of $\mathcal{I}(\xi, \eta)$ (see theorem 3.1). If η is allowed to range in a neighbourhood of $\eta = 0$, the regions shaded in the figures may be derived from (28) setting $\xi = \rho^2/2$, i. e. from: $dI_1 = -a\rho^2/2 + \eta^2 + \mathrm{O}_3(\rho, \eta)$ and the normal character of the tori follows from the characteristic exponents (31).

3.4. **Parametrization of the invariant manifolds of the hyperbolic periodic orbits.** We recall that when $\sigma_1 < 0$ in (22) (i. e., when $dI_1 < 0$ with $|I_1|$ small) then the orbit $\mathcal{M}_{I_1}$ given by the parametrization (21) is a hyperbolic periodic orbit of the Hamiltonian equations of the normal form $Z^{(r)}$. So, one may use $Z^{(r)}$ to get parametrizations of the stable and unstable invariant manifolds of this orbits. If we consider a fixed I_1 such that $\sigma_1 < 0$, then the corresponding stable and unstable manifolds of $\mathcal{M}_{I_1}$ are three dimensional and can be obtained by setting the values of the first integrals in (20) to the ones of $\mathcal{M}_{I_1}$. If we write $\mathcal{I}_2$ and $\mathcal{I}_3$ in the coordinates (23) we have,

$$\mathcal{I}_2 = I_2, \qquad \mathcal{I}_3 = qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2).$$

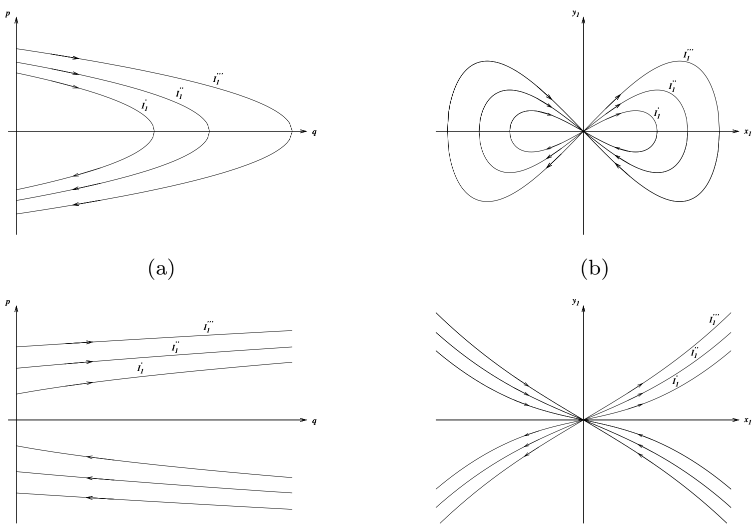
So, the invariant manifolds of $\mathcal{M}_{I_1}$ are defined by

$$I_2 = 0, \qquad qp^2 + \mathcal{Z}_r(q, I_1, 0) = \mathcal{Z}_r(0, I_1, 0),$$

obtaining:

$$p = \pm \left\{ \frac{1}{q} \left[\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0) \right] \right\}^{1/2} = \pm \left[-\frac{a}{2}q - dI_1 + \mathrm{O}_2(q, I_1) \right]^{1/2}.$$

The choice + corresponds to the unstable manifold and the choice − to the stable one. Going back to the rectangular coordinates through (23), we obtain the



(a) (b)

(c) (d)

FIGURE 2. Invariant manifolds of the hyperbolic periodic orbits for three different values of I_1: $I_1''' < I_1'' < I_1' < 0$. Figures (a) and (b), corresponding to $a > 0$, are the projections of the invariant manifolds on the planes (q, p) and (x_1, y_1) respectively. The same, but for $a < 0$ is plotted in figures (c) and (d).

following parametrization of the manifolds:

$$x_1 = \sqrt{2q}\cos\theta_2, \qquad y_1 = \pm\left[2(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))\right]^{1/2}\cos\theta_2,$$
$$x_2 = -\sqrt{2q}\sin\theta_2, \qquad y_2 = \mp\left[2(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))\right]^{1/2}\sin\theta_2.$$

Alternatively, the invariant manifolds can be given as graphs:

$$\begin{aligned} y_i &= \pm x_i\left\{\frac{2}{x_1^2 + x_2^2}\left[\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r\left(\frac{1}{2}\left(x_1^2 + x_2^2\right), I_1, 0\right)\right]\right\}^{1/2} \\ &= \pm x_i\left[-\frac{a}{4}(x_1^2 + x_2^2) - dI_1 + \Gamma\right]^{1/2}, \end{aligned}$$

where Γ stands for the terms of (adapted) degree at least 3. These parametrizations are represented in figures 2(a)–(d) for three different (negative) values of the action I_1 and according to the sign of a (see the details in the caption).

However, the range of available parameters (q, I_1) is restricted by the condition that the expressions inside the square roots must be positive. Introducing $F(q, I_1)$ through

$$F(q, I_1) := \frac{1}{q}\left[\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0)\right],$$

clearly, $F(q, I_1) \geq 0$ must hold and we notice that $F(0, 0) = 0$, $\partial_1 F(0, 0) = -a/2 \neq 0$ and $\partial_2 F(0, 0) = -d \neq 0$. Thus, the boundary of this domain can be (locally) expressed as function of I_1 or q.

3.5. **Parametrization of the invariant manifolds of the hyperbolic 2D-invariant tori.** In the inverse case (when $a < 0$), we have shown that for certain range of the parameters (ξ, η) the 2-dimensional bifurcated invariant torus $\mathcal{T}_{\xi,\eta}$ given in (26) is normally hyperbolic. More precisely, it happens for the values of $\xi > 0$ and η such that

$$4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) < 0$$

holds (see theorem 3.1 and proposition 3.1). For these tori we can also compute their stable and unstable manifold which, for any given torus, have dimension three. Again, they are implicitly defined by fixing the values of the first integrals in (20):

$$\mathcal{I}_1 = \mathcal{I}(\xi, \eta), \quad \mathcal{I}_2 = 2\xi\eta, \quad \mathcal{I}_3 = \xi\eta^2 + \mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta).$$

Using the coordinates (23) we obtain the following expression for p:

$$\begin{aligned} p &= \pm\sqrt{F(q, \xi, \eta)} \qquad (32)\\ &= \pm\left\{\frac{1}{q}\left[\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) - \mathcal{Z}_r(q, \mathcal{I}(\xi, \eta), 2\xi\eta) + \xi^2\eta^2\left(\frac{1}{\xi} - \frac{1}{q}\right)\right]\right\}^{1/2} \end{aligned}$$

and the corresponding manifold will be stable if $p(q-\xi) < 0$ or unstable if $p(q-\xi) > 0$ (see figure 3(a)). Of course, using the expressions (23) we can go back to the original normal form coordinates. As for the hyperbolic periodic orbits, we can also see these manifolds as graphs, so that

$$\begin{aligned} y_1 &= 2\xi\eta\frac{x_2}{x_1^2 + x_2^2} \pm x_1\left[F\left(\frac{1}{2}(x_1^2 + x_2^2), \xi, \eta\right)\right]^{1/2},\\ y_2 &= -2\xi\eta\frac{x_1}{x_1^2 + x_2^2} \pm x_2\left[F\left(\frac{1}{2}(x_1^2 + x_2^2), \xi, \eta\right)\right]^{1/2}. \end{aligned}$$

Moreover, if we use the equation defining $\mathcal{I}(\xi, \eta)$ (see (28)) we can make more clear the expression of $F(q, \xi, \eta)$. Indeed, we can expand it in powers of $q - \xi$, obtaining:

$$\begin{aligned} \mathcal{Z}_r(q, \mathcal{I}(\xi, \eta), 2\xi\eta) - \mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) &= \eta^2(q - \xi)\\ &\quad + \frac{1}{2}\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta)(q - \xi)^2\\ &\quad + G(q, \xi, \eta)(q - \xi)^3, \end{aligned}$$

where $G(q, \xi, \eta)(q-\xi)^3$ stands for the complementary term in the Taylor expansion, and thus

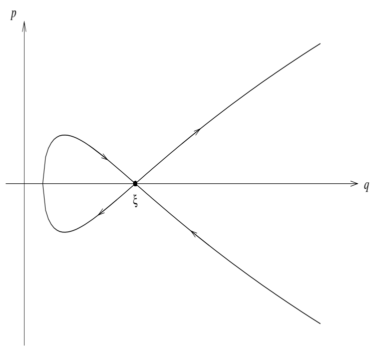
$$F = -\frac{(q - \xi)^2}{4q^2}g(q, \xi, \eta)$$

with

$$\begin{aligned} g(q, \xi, \eta) &= 4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\zeta), 2\xi\eta)\\ &\quad + 2(q - \xi)\left[\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\zeta), 2\xi\eta) + 2qG(q, \zeta)\right]. \end{aligned}$$

Using the expression above for F, the condition on the torus $\mathcal{T}_{\xi,\eta}$ to be hyperbolic appears in a natural form: as in equation (32) we need $F(q, \xi, \eta) \geq 0$. Thus, whenever $4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) < 0$, this condition is fulfilled provided $|q - \xi|$ is small. In figure 3 we plot, for some given values of the parameters (ξ, η), the invariant manifolds of the hyperbolic tori. The projections are displayed on the plane (q, p) –in figure 3(a)– and on the plane (y_1, y_2) –in figure 3(b)–.

Moreover, if we want to characterize the range of available parameters in (32), we have to study the solutions of $g(q, \xi, \eta) = 0$. Actually, as $g(0, 0, 0) = 0$ and



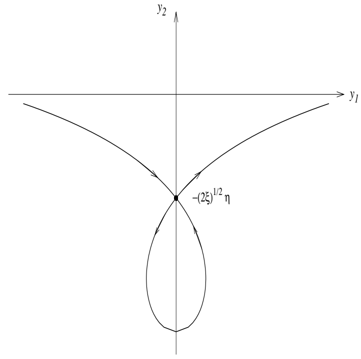


FIGURE 3. Invariant manifolds of the hyperbolic invariant tori corresponding to (ξ, η). In figure (b) we set $x_2 = 0$ and $\eta > 0$ is assumed. The position of the hyperbolic invariant tori is $(q, p) = (\xi, 0)$ in (a) and $(y_1, y_2) = (0, -\eta\sqrt{2\xi})$ in (b). In both cases is marked with a dot.

the derivative $\partial_1 g(0,0,0) = 2a \neq 0$, we can give the solutions of $g(q, \xi, \eta) = 0$ by writing $q = f(\xi, \eta)$, so that the boundary of values of q in the parameter space of F is given (locally around $(\xi, \eta) = (0,0)$) as $q \geq f(\xi, \eta)$.

3.6. **Computation of 3D-invariant tori.** The $3D$-invariant tori of the normal form (16) can be obtained from periodic orbits of the 1-degree of freedom Hamiltonian system given by:

$$H(q, p; I_1, I_2) = qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2), \tag{33}$$

where I_1 and I_2 have to be treated as parameters (see (24)). Thus, given a couple of values I_1 and I_2 (fixed), let $(\widetilde{q}(\theta_3), \widetilde{p}(\theta_3))$ be a 2π-periodic parametrization of a periodic orbit of (33) such that $\dot{\theta}_3 = \widetilde{\omega}_3$. Then, the dynamics of the corresponding $3D$-invariant torus of (24) can be obtained by direct integration of the expressions,

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + \partial_2 \mathcal{Z}_r(\widetilde{q}(\widetilde{\omega}_3 t), I_1, I_2), \\ \dot{\theta}_2 &= \omega_2 + \frac{I_2}{2\widetilde{q}(\widetilde{\omega}_3 t)} + \partial_3 \mathcal{Z}_r(\widetilde{q}(\widetilde{\omega}_3 t), I_1, I_2), \end{aligned}$$

being the vector of intrinsic frequencies of this torus $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$, with $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ defined by

$$\begin{aligned} \widetilde{\omega}_1 &= \omega_1 + \langle \partial_2 \mathcal{Z}_r(\widetilde{q}(\theta_3), I_1, I_2) \rangle, \\ \widetilde{\omega}_2 &= \omega_2 + \left\langle \frac{I_2}{2\widetilde{q}(\theta_3)} \right\rangle + \langle \partial_3 \mathcal{Z}_r(\widetilde{q}(\theta_3), I_1, I_2) \rangle, \end{aligned}$$

where $\langle \cdot \rangle$ denotes the average with respect to the angle θ_3 (of course, we need $\widetilde{\omega}_1$, $\widetilde{\omega}_2$ and $\widetilde{\omega}_3$ to be independent frequencies in order to have a legitimate $3D$-torus).

To discuss the range of parameters for which one obtains periodic orbits of (33), we point out that these orbits can be obtained implicitly as energy levels of the system, $\{H(q,p;I_1,I_2)=h\}$, for suitable values of h. The extremal values for the interval of allowed values of h (for any given I_1 and I_2), correspond to the ones of the critical points of H.

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