34 Pels sòlids següents, calculeu els moment d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a u.

(a) Calculeu I_Z pel sòlid de revolució d'alçada H i radi de la base R donat per $x^2+y^2 \le \frac{R^2}{H^2} z^2 \ (0 \le z \le H)$

$$D = \left\{ x^2+y^2 \le \frac{R^2}{H^2} z^2,\ 0 \le z \le H \right\} \xrightarrow[\text{Cilíndriques}]{\text{Coord.}} D^* = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le H \\ 0 \le r \le Rz/H \end{array} \right\}$$

$$I_Z = \iiint_D (x^2+y^2)\, \rho(x,y,z)\, dx\,dy\,dz = \iiint_{D^*} r^2 \cdot \underbrace{r}_{\text{Jacobià}} \cdot dr\,d\theta\,dz =$$

$$= \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_0^H \left(\int_0^{Rz/H} r^3\,dr\right) dz\right) = 2\pi \int_0^H \left[\frac{r^4}{4}\right]_{r=0}^{r=Rz/H} dz = \frac{\pi}{2} \frac{R^4}{H^4} \int_0^H z^4\,dz =$$

$$= \frac{\pi}{2} \frac{R^4}{H^4} \left[\frac{z^5}{5}\right]_{z=0}^{z=H} = \frac{1}{10} \pi H R^4$$

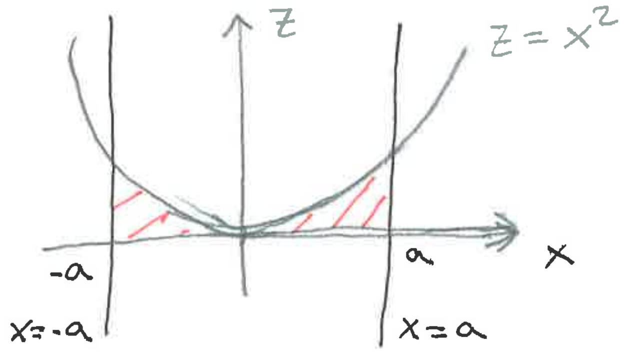
(b) Calculeu I_Z pel sòlid limitat per dos cilindres d'alçada h, $a^2 \le x^2+y^2 \le b^2,\ 0 \le z \le h$.

$$D = \{ a^2 \le x^2+y^2 \le b^2,\ 0 \le z \le h \} \xrightarrow[\text{Cilíndriques}]{\text{Coord.}} D^* = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ a \le r \le b \\ 0 \le z \le h \end{array} \right\}$$

$$I_Z = \iiint_D (x^2+y^2)\, \rho(x,y,z)\, dx\,dy\,dz = \iiint_{D^*} r^2 \cdot r\,dr\,d\theta\,dz =$$

$$= \left(\int_a^b r^3\,dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^h dz\right) = \frac{b^4-a^4}{4} \cdot 2\pi \cdot h = \frac{b^4-a^4}{2} \pi h$$

(c) Calculeu I_Z pel sòlid limitat pel paraboloide $z = x^2+y^2$ i el cilindre $x^2+y^2 = a^2$, $(z \ge 0)$.



- El dibuix correspon a fer la secció pel pla $y=0$ del domini.
- El domini, D, s'obté fent girar la part enombrejada entorn de l'eix Z.

$$D=\{0\le z\le x^2+y^2,\ 0\le x^2+y^2\le a^2\}\ \xrightarrow[\text{cilíndriques}]{\text{coord.}}\ D^*=\begin{Bmatrix}0\le\theta\le 2\pi\\ 0\le r\le a\\ 0\le z\le r^2\end{Bmatrix}$$

$$I_z=\iiint_D (x^2+y^2)\rho(x,y,z)\,dx\,dy\,dz=\iiint_{D^*} r^2\cdot r\,dr\,d\theta\,dz=$$

$$=\left(\int_0^{2\pi}d\theta\right)\cdot\left(\int_0^a\left(\int_0^{r^2}r^3dz\right)dr\right)=2\pi\int_0^a r^3\cdot r^2dr=2\pi\left[\frac{r^6}{6}\right]_{r=0}^{r=a}=\pi\frac{a^6}{3}$$

(d) Calculeu I_x, I_y i I_z pel sòlid tancat pel paraboloide $z=x^2+y^2$ i el pla $z=a$ $(a>0)$.

$$D=\{0\le z\le a,\ 0\le x^2+y^2\le z\}\ \xrightarrow[\text{cilíndriques}]{\text{coord.}}\ D^*=\begin{Bmatrix}0\le\theta\le 2\pi\\ 0\le z\le a\\ 0\le r^2\le z\end{Bmatrix}$$

D domini de revolució entorn de l'eix z: per tant, per simetria tenim $I_x=I_y$.

$$I_z=\iiint_D (x^2+y^2)\rho(x,y,z)\,dx\,dy\,dz=\iiint_{D^*} r^2\cdot r\,dr\,d\theta\,dz=$$

$$=\left(\int_0^{2\pi}d\theta\right)\cdot\left(\int_0^a\left(\int_0^{\sqrt z}r^3dr\right)dz\right)=2\pi\int_0^a\left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt z}dz=\frac{\pi}{2}\int_0^a z^2dz=\frac{\pi}{6}a^3$$

$$I_x=\iiint_D (y^2+z^2)\rho(x,y,z)\,dx\,dy\,dz=\iiint_{D^*}(r^2\sin^2\theta+z^2)\,r\,dr\,d\theta\,dz=$$

$$=\underbrace{\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)}_{=\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta=\pi}\cdot\underbrace{\left(\int_0^a\left(\int_0^{\sqrt z}r^3dr\right)dz\right)}_{=a^3/12}+\underbrace{\left(\int_0^{2\pi}d\theta\right)}_{=2\pi}\cdot\underbrace{\left(\int_0^a\left(\int_0^{\sqrt z}z^2r\,dr\right)dz\right)}_{=\int_0^a z^2\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt z}dz=\int_0^a\frac{z^3}{2}dz=\frac{a^4}{8}}=$$

$$=\pi\frac{a^3}{12}+2\pi\cdot\frac{a^4}{8}=\frac{\pi}{12}a^3(1+3a)$$

(e) Calculeu I_x, I_y i I_z pel cilindre $x^2+y^2\le R^2$, $-h\le z\le h$.

$$D=\{x^2+y^2\le R^2,\ -h\le z\le h\}\ \xrightarrow[\text{cilíndriques}]{\text{coord.}}\ D^*=\begin{Bmatrix}0\le\theta\le 2\pi\\ 0\le r\le R\\ -h\le z\le h\end{Bmatrix}$$

Per simetria de revolució és clar $I_x=I_y$.

$$I_x=\iiint_{D^*}(r^2\sin^2\theta+z^2)\,r\,dr\,d\theta\,dz=\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_0^R r^3dr\right)\left(\int_{-h}^h dz\right)+$$

$$+\left(\int_0^{2\pi}d\theta\right)\left(\int_0^R r\,dr\right)\left(\int_{-h}^h z^2dz\right)=\pi\frac{R^4}{4}2h+2\pi\frac{R^2}{2}\frac{2}{3}h^3=\frac{\pi hR^2}{6}(3R^2+4h^2)$$