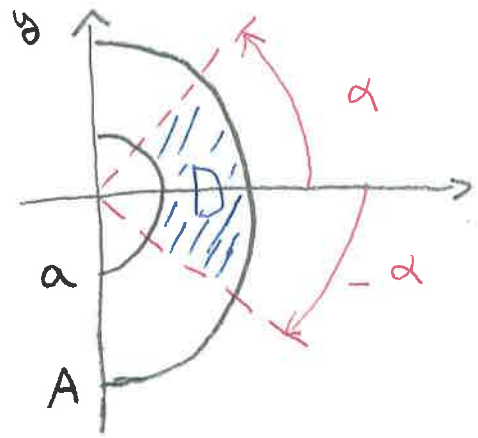
(25) Trobeu el centre de masses de les regions planes següents amb les densitats que s'indiquen.

(a) Sector pla definit per una corona de radi interior a i radi exterior A, un angle d'obertura 2α i que és simètrica respecte de l'eix x positiu, suposant densitat constant $\rho(x,y)=1$.



• $GM = (\bar{x}, \bar{y}) \underset{\uparrow}{=} (\bar{x}, 0)$

Per simetria D i ρ.

• $\bar{x} = \dfrac{1}{m(D)} \iint_D x\,\rho(x,y)\,dx\,dy$

• $m(D) = \iint_D \rho(x,y)\,dx\,dy = \iint_D dx\,dy = \text{Àrea}(D) =$

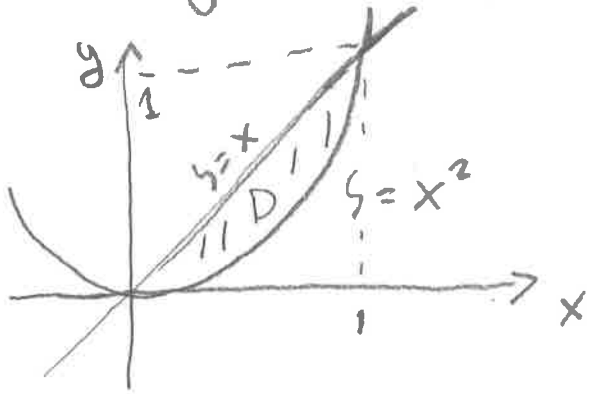
$$= \underbrace{(\pi A^2 - \pi a^2)}_{\text{àrea corona circular}} \underbrace{\frac{2\alpha}{2\pi}}_{\text{només ens queden un tros d'angle } 2\alpha} = \alpha(A^2 - a^2)$$

• $\bar{x} = \dfrac{1}{m(D)} \iint_D x\,dx\,dy = \left\{ \begin{array}{l} \text{Coord. polars} \\ D^* = \left\{ \begin{array}{l} a \le r \le A \\ -\alpha \le \theta \le \alpha \end{array} \right\} \end{array} \right\} =$

$$= \frac{1}{m(D)} \iint_{D^*} r\cos\theta \cdot r\,dr\,d\theta = \frac{1}{m(D)} \left(\int_a^A r^3\,dr \right) \left(\int_{-\alpha}^{\alpha} \cos\theta\,d\theta \right) =$$

$$= \frac{1}{\alpha(A^2-a^2)} \cdot \frac{A^3-a^3}{3} \cdot 2\sin\alpha = \frac{2}{3}\,\frac{\sin\alpha}{\alpha}\,\frac{A^3-a^3}{A^2-a^2}$$

(b) Regió entre $y=x^2$ i $y=x$ amb $\rho(x,y)=x+y$.



• $CM = (\bar{x}, \bar{y})$

• $D = \left\{ \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le x \end{array} \right\}$

• $m(D) = \iint_D \rho(x,y)\,dx\,dy = \iint_D (x+y)\,dx\,dy = \{\text{Fubini}\} =$

$$= \int_0^1 \left(\int_{x^2}^{x} (x+y)\,dy \right) dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx =$$

$$= \int_0^1 \left(x^2 + \frac{x^2}{2} - \left(x^3 + \frac{x^4}{2} \right) \right) dx = \left[\frac{3}{2}\frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{10} \right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{3}{20}.$$

• $\bar{x} = \frac{1}{m(D)} \iint_D x\,\rho(x,y)\,dx\,dy = \frac{20}{3} \iint_D x(x+y)\,dx\,dy =$

$$= \frac{20}{3} \int_0^1 \left(\int_{x^2}^{x} (x^2 + xy)\,dy \right) dx = \frac{20}{3} \int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_{y=x^2}^{y=x} dx =$$

$$= \frac{20}{3} \int_0^1 \left(x^3 + \frac{x^3}{2} - \left(x^4 + \frac{x^5}{2} \right) \right) dx = \frac{20}{3} \left[\frac{3}{2}\frac{x^4}{5} - \frac{x^5}{5} - \frac{x^6}{12} \right]_{x=0}^{x=1} = 1/9$$

• $\bar{y} = \frac{1}{m(D)} \iint_D y\,\rho(x,y)\,dx\,dy = \frac{20}{3} \iint_D y(x+y)\,dx\,dy =$

$$= \frac{20}{3} \int_0^1 \left(\int_{x^2}^{x} \left(yx + \frac{y^2}{2} \right) dy \right) dx = \frac{20}{3} \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{6} \right]_{y=x^2}^{y=x} dx =$$

$$= \frac{20}{3} \int_0^1 \left(\frac{x^3}{2} + \frac{x^3}{6} - \left(\frac{x^5}{2} + \frac{x^6}{6} \right) \right) dx = \frac{20}{3} \left[\frac{2}{3}\frac{x^4}{4} - \frac{x^6}{12} - \frac{x^7}{42} \right]_{x=0}^{x=1} = \frac{25}{62}$$