(21) Calculeu els volums dels dominis $B \subset \mathbb{R}^3$ definits en coordenades esfèriques $x = r\cos\varphi\cos\theta$, $y = r\cos\varphi\sin\theta$, $r = \sin\varphi$ $(0 < \theta < 2\pi,$ $-\frac{\pi}{2} < \varphi < \frac{\pi}{2})$ que s'indiquen tot seguit.

(a) B: domini tallat sobre la bola $r \leq a$ pel con $\alpha \leq \varphi \leq \frac{\pi}{2}$ $(a > 0,\ 0 < \alpha < \frac{\pi}{2})$.

$$\text{Volum}(B) = \iiint_B 1\, dx\, dy\, dz \underset{\uparrow}{=} \iiint_{B^*} \underbrace{r^2\cos\varphi}_{\text{Jacobià}}\, dr\, d\theta\, d\varphi =$$

B^* domini en esfèriques

$$= \int_0^a \int_0^{2\pi} \int_\alpha^{\pi/2} r^2 \cos\varphi\, d\varphi\, d\theta\, dr = 2\pi \cdot \left[\frac{r^3}{3}\right]_{r=0}^{r=a} \cdot \left[\sin\varphi\right]_{\varphi=\alpha}^{\varphi=\pi/2} = \frac{2\pi a^3}{3}(1-\sin\alpha)$$

(b) B volum tancat per l'esfera deformada definida per $r = 1 + 0.2\sin(8\theta)\cdot\sin\varphi$ (sòlids d'aquesta mena s'utilitzen com a models de tumors).

Domini B en esfèriques: $0 \leq \theta \leq 2\pi$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$,
$0 \leq r \leq 1 + 0.2\cdot\sin(8\theta)\cdot\sin\varphi$.

$$\text{Volum}(B) = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^{1+0.2\cdot\sin(8\theta)\cdot\sin\varphi} r^2\cos\varphi\, dr\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\varphi \left[\frac{r^3}{3}\right]_{r=0}^{r=1+0.2\cdot\sin(8\theta)\sin\varphi} d\varphi \cdot d\theta =$$

$$= \frac{1}{3}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\varphi\,(1+0.2\sin(8\theta)\sin\varphi)^3\, d\varphi\, d\theta =$$

$$= \frac{1}{3}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos\varphi\left(1 + 0.6\sin(8\theta)\sin\varphi + 0.12\sin^2(8\theta)\sin^2\varphi + 8\cdot10^{-3}\sin^3(8\theta)\sin^3\varphi\right) d\varphi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} (\cos\varphi + 0.12\sin^2(8\theta)\cdot\sin^2\varphi\cos\varphi)\, d\varphi\, d\theta,$$

on usem que no costa gens veure que $\int_0^{2\pi} \sin(8\theta)\, d\theta = 0$ i que també $\int_0^{2\pi} \sin^3(8\theta)\,d\theta = 0$ (aquesta darrera la podem pendre com una indicació si cal). Així:

$$I = \frac{2\pi}{3}\left[\sin\varphi\right]_{\varphi=-\pi/2}^{\varphi=\pi/2} + \frac{0.12}{3}\int_0^{2\pi}\sin^2(8\theta)\,d\theta \cdot \left[\frac{\sin^3\varphi}{3}\right]_{\varphi=-\pi/2}^{\varphi=\pi/2} =$$

$$= \frac{4\pi}{3} + \frac{0.08}{3}\pi = \frac{4.08\pi}{3}$$, on usem que:

$$\int_0^{2\pi}\sin^2(8\theta)\,d\theta = \int_0^{2\pi}\frac{1-\cos(16\theta)}{2}\,d\theta = \pi$$ (el cosinus no contribueix).

(c) Anàlogament, calculeu la integral triple

$$I = \iiint_B \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$$ on B és la regió del primer octant de $\mathbb{R}^3$ acotada pels cons $\varphi = \frac{\pi}{4}$ i $\varphi = \arctan(2)$ i l'esfera $r = \sqrt{6}$.

El domini B en esfèriques és: $0 \leq \theta \leq 2\pi$, $\frac{\pi}{4} \leq \varphi \leq \arctan(2)$
$0 \leq r \leq \sqrt{6}$.

Així:

$$I = \int_0^{\sqrt{6}}\int_0^{2\pi}\int_{\pi/4}^{\arctan(2)} \frac{1}{r} r^2\cos\varphi\,d\varphi\,d\theta\,dr = 2\pi\int_0^{\sqrt{6}} r\,dr \cdot \int_{\pi/4}^{\arctan(2)}\cos\varphi\,d\varphi =$$

$$= 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{6}} \cdot \left[\sin\varphi\right]_{\varphi=\pi/4}^{\varphi=\arctan(2)} = 6\pi\left(\sin(\arctan(2)) - \sin(\pi/4)\right) =$$

$$= 6\pi\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)$$

Ja que podem calcular $\sin(\arctan(2))$ usant que:

$$\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} = \frac{\sin^2(x)}{1-\sin^2(x)} \Rightarrow \tan^2(x)\left(1-\sin^2(x)\right) = \sin^2(x) \Rightarrow$$

$$\Rightarrow \sin^2(x) = \frac{\tan^2(x)}{1+\tan^2(x)}$$. Fent $x = \arctan(2)$ obtenim:

$$\sin^2(\arctan(2)) = \frac{2^2}{1+2^2} = \frac{4}{5}.$$