(19) Useu coordenades cilíndriques per calcular les següents integrals triples.

Coord. cilíndriques: $(x,y,z) = (r\cos\theta, r\sin\theta, z)$, $r\in[0,+\infty)$, $\theta\in(0,2\pi)$, $z\in\mathbb{R}$

$J(r,\theta,z) = r.$

(a) $I = \iiint_B \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : \sqrt{x^2+y^2}\le z\le 4\}$

El domini B en coord. cilíndriques ve donat per:

$\sqrt{x^2+y^2}\le z\le 4 \Rightarrow r\le z\le 4$ i $0\le\theta\le 2\pi$. Així:

$$I = \int_0^{2\pi}\int_0^4\int_0^z \sqrt{r^2+z^2}\, r\, dr\, dz\, d\theta = 2\pi\int_0^4 \left[\frac{(r^2+z^2)^{3/2}}{\frac{3}{2}\cdot 2}\right]_{r=0}^{r=z} dz = \frac{2\pi}{3}\int_0^4 [(2z^2)^{3/2}-(z^2)^{3/2}]\, dz =$$

$$= \frac{2\pi}{3}(2\sqrt{2}-1)\int_0^4 z^3\, dz = \frac{2\pi}{3}(2\sqrt{2}-1)\left[\frac{z^4}{4}\right]_{z=0}^{z=4} = \frac{2\pi}{3}(2\sqrt{2}-1)\frac{4^4}{4} = \frac{128}{3}\pi(2\sqrt{2}-1)$$

(b) $I = \iiint_B z e^{-(x^2+y^2)}\, dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : z^2-1\le x^2+y^2\le z^2/2,\ z\ge 0\}$

El domini B en cilíndriques ve donat per:

$z^2-1\le r^2\le z^2/2 \Rightarrow \sqrt{z^2-1}\le r\le z/\sqrt{2}$.

Per a que aquesta condició sobre r tingui sentit cal que

$z^2-1\le z^2/2 \Leftrightarrow z^2\le 2$

$z^2-1\ge 0 \Leftrightarrow z^2\ge 1$

Per tant la condició sobre z és $1\le z\le\sqrt{2}$.

Per la seva banda $0\le\theta\le 2\pi$. Així:

$$I = \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\sqrt{z^2-1}}^{z/\sqrt{2}} z e^{-r^2} r\, dr\, dz\, d\theta = 2\pi\int_1^{\sqrt{2}} z\left[\frac{-e^{-r^2}}{2}\right]_{r=\sqrt{z^2-1}}^{r=z/\sqrt{2}} dz =$$

$$= \pi\int_1^{\sqrt{2}} z(e^{-(z^2-1)} - e^{-z^2/2})\, dz = \pi e\int_1^{\sqrt{2}} z e^{-z^2} dz - \pi\int_1^{\sqrt{2}} z e^{-z^2/2} dz =$$

$$= \pi e\left[\frac{e^{-z^2}}{-2}\right]_{z=1}^{z=\sqrt{2}} - \pi\left[\frac{e^{-z^2/2}}{-1}\right]_{z=1}^{z=\sqrt{2}} = \frac{\pi e}{2}(e^{-1}-e^{-2}) + \pi(e^{-1}-e^{-1/2}) =$$

$$= \frac{\pi}{2} + \frac{\pi}{2e} - \frac{\pi}{\sqrt{e}} + \frac{\pi}{e} = \frac{\pi}{2} + \frac{3\pi}{2e} - \frac{\pi}{\sqrt{e}}$$

(c) $I=\iiint_B (x+y-2z)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le z^2,\ 0\le z\le 3\}$

El domini B en cilíndriques ve donat per:

$x^2+y^2\le z^2 \Rightarrow r^2\le z^2 \underset{(r\ge0;\ z\ge0)}{\Rightarrow} 0\le r\le z$, Per altra banda $0\le z\le 3$, $0\le\theta\le 2\pi$.

Així: $I=\int_0^{2\pi}\int_0^3\int_0^z (r\cos\theta+r\sin\theta-2z)\,r\,dr\,dz\,d\theta=$

$=\int_0^{2\pi}\int_0^3\left[\frac{r^3}{3}\cos\theta+\frac{r^3}{3}\sin\theta-2z\frac{r^2}{2}\right]_{r=0}^{r=z}dz\,d\theta=$

$=\int_0^3\int_0^{2\pi}\left(\frac{z^3}{3}\cos\theta+\frac{z^3}{3}\sin\theta-z^3\right)d\theta\,dz\underset{\uparrow}{=}-2\pi\int_0^3 z^3\,dz=-2\pi\left[\frac{z^4}{4}\right]_{z=0}^{z=3}=-\frac{81\pi}{2}$

$\int_0^{2\pi}\cos\theta\,d\theta=\int_0^{2\pi}\sin\theta\,d\theta=0.$

(d) $I=\iiint_B (x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 3z\le 9\}$

El domini B en cilíndriques ve donat per:

$x^2+y^2\le 3z\le 9 \Rightarrow r^2\le 3z\le 9 \underset{(r\ge0)}{\Rightarrow} 0\le z\le 3,\ 0\le r\le\sqrt{3z}$ i $0\le\theta\le 2\pi$

$I=\int_0^{2\pi}\int_0^3\int_0^{\sqrt{3z}} r^2\cdot r\,dr\,dz\,d\theta=2\pi\int_0^3\left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{3z}}dz=\frac{9}{2}\pi\int_0^3 z^2\,dz=\frac{9}{2}\pi\left[\frac{z^3}{3}\right]_{z=0}^{z=3}=\frac{81\pi}{2}$

(e) $I=\iiint_B z\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 6,\ x^2+y^2\le z,\ z\ge0\}$

El domini B en cilíndriques ve donat per:

$x^2+y^2+z^2\le 6 \Rightarrow r^2+z^2\le 6 \Rightarrow z^2\le 6-r^2 \Rightarrow z\le\sqrt{6-r^2}$

$x^2+y^2\le z \Rightarrow r^2\le z$ (en particular cal $z\ge0$)

Així doncs cal: $0\le r\le\sqrt{2}$, $r^2\le z\le\sqrt{6-r^2}$, $0\le\theta\le 2\pi$.

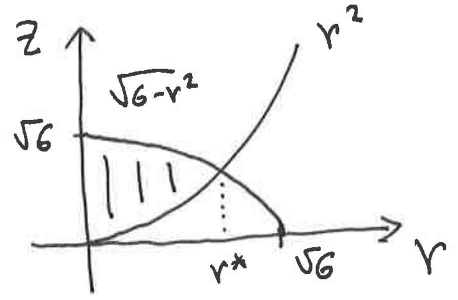
(per tal que $\sqrt{6-r^2}\ge r^2$).

$I=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} z\cdot r\,dz\,dr\,d\theta=2\pi\int_0^{\sqrt{2}} r\left[\frac{z^2}{2}\right]_{z=r^2}^{z=\sqrt{6-r^2}}dr=\pi\int_0^{\sqrt{2}} r(6-r^2-r^4)\,dr=$

$=\pi\left[\frac{6r^2}{2}-\frac{r^4}{4}-\frac{r^6}{6}\right]_{r=0}^{r=\sqrt{2}}=\pi\left(3\cdot2-\frac{2^2}{4}-\frac{2^3}{6}\right)=\frac{11}{3}\pi.$

Fixeu-vos que per justificar que la condició $r^2 \leq z \leq \sqrt{6-r^2}$ determina un rang de valors de z no buit, cal demanar que els valors de r triats verifiquin $r^2 \leq \sqrt{6-r^2}$.

Gràficament:



(|||) Correspon a als valors de z entre r^2 i $\sqrt{6-r^2}$.

r^* és solució de $r^2 = \sqrt{6-r^2} \iff$

$\iff r^4 + r^2 - 6 = 0$

$\iff r^2 = \dfrac{-1 \pm \sqrt{1+24}}{2} = \dfrac{-1 \pm 5}{2} = \left\langle \begin{matrix} 2 \\ -3 \end{matrix} \right.$

Així doncs, $r_*^2 = 2 \Rightarrow r_* = \sqrt{2}$.