(18) Calculeu les integrals dobles següents mitjançant el canvi de variables que s'indica en cada cas.

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} (f\circ\varphi)(u,v)\,\left|J_\varphi(u,v)\right|\,du\,dv \quad , \text{ on}$$

$$\begin{aligned}\varphi : D^* &\longrightarrow D \\ (u,v) &\longmapsto (x,y)=\varphi(u,v)\end{aligned} \quad \text{canvi de variables i } J_\varphi(u,v) = \det(D\varphi(u,v)) \text{ Jacobià.}$$

(a) $I = \iint_D xy\,dx\,dy$, $D = \{(x,y)\in\mathbb{R}^2 : 6 \le 2y - x \le 12 ,\ 0\le x\le 4\}$ fent

$x = 4u$ i $y = 2u + 3v$.

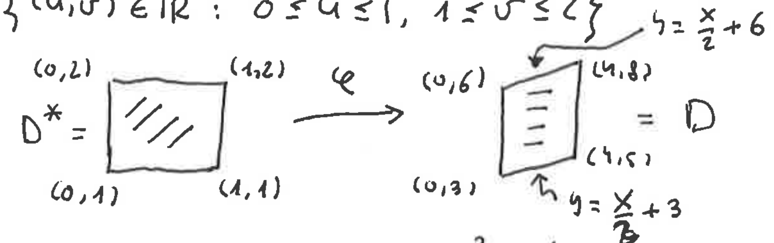
És clar $\varphi(u,v) = (4u, 2u+3v)$ i $D\varphi(u,v) = \begin{pmatrix} 4 & 0 \\ 2 & 3\end{pmatrix}$, $J_\varphi(u,v) = 12$.

El domini D^* ve donat per:

$$\begin{cases} 6 \le 2y - x \le 12 \Rightarrow 6 \le 2(2u+3v) - 4u \le 12 \Rightarrow 6 \le 6v \le 12 \Rightarrow 1 \le v \le 2 \\ 0 \le x \le 4 \Rightarrow 0 \le 4u \le 4 \Rightarrow 0 \le u \le 1.\end{cases}$$

Llavors $D^* = \{(u,v)\in\mathbb{R}^2 : 0\le u\le 1,\ 1\le v\le 2\}$

Gràficament :



$$I = \iint_{D^*} 12\cdot 4u(2u+3v)\,du\,dv = 48\int_1^2\left(\int_0^1 (2u^2 + 3uv)\,du\right)dv =$$

$$= 48\int_1^2 \left[\frac{2u^3}{3} + \frac{3u^2 v}{3}\right]_{u=0}^{u=1} dv = 48\int_1^2\left(\frac{2}{3} + v\right)dv = 48\left(\frac{2}{3} + \left[\frac{v^2}{2}\right]_{v=1}^{v=2}\right) =$$

$$= 48\left(\frac{2}{3} + \frac{4}{2} - \frac{1}{2}\right) = \frac{48}{6}(4+9) = 624.$$

(c) $I = \iint_D \frac{dx\,dy}{(x+y)^{n+1}}$, $D = \{(x,y)\in\mathbb{R}^2 : 1\le x+y\le 2,\ x\ge 0,\ y\ge 0\}$, fent

$u = x+y$ i $v = y$.

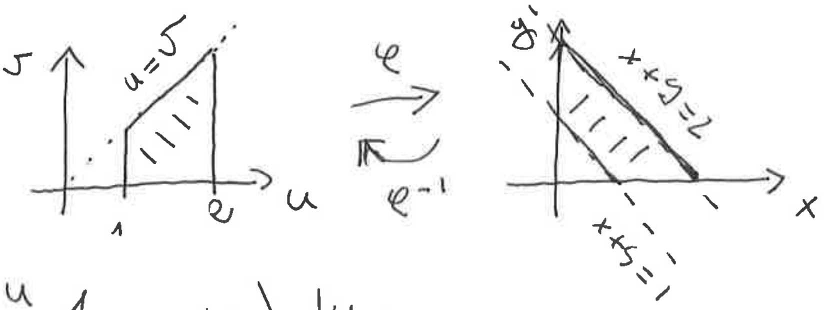
Atenció: Ara és $\varphi^{-1}(x,y) = (x+y, y)$, $D\varphi^{-1}(x,y) = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$, $J_{\varphi^{-1}}(x,y) = 1$

Per tant: $J_\varphi(u,v) = \frac{1}{J_{\varphi^{-1}}(x,y)} = \frac{1}{1} = 1$

El domini D^* ve donat per:

$$\left.\begin{aligned} &1 \le x+y \le 2 \Rightarrow 1 \le u \le 2 \\ &y \ge 0 \Rightarrow v \ge 0 \\ &x \ge 0 \Rightarrow x = u - y = u - v \ge 0 \Rightarrow u \ge v\end{aligned}\right\} \quad D^* = \{(u,v)\in\mathbb{R}^2 : 1\le u\le 2,\ 0\le v\le u\}$$

Gràficament:



$$I = \int_1^2 \left(\int_0^u \frac{1}{u^{n+1}}\, dv \right) du =$$

$$= \int_1^2 \frac{u}{u^{n+1}}\, du = \int_1^2 u^{-n}\, du = \left[\frac{u^{-n+1}}{-n+1} \right]_{u=1}^{u=2} = \frac{1}{1-n}\left(2^{1-n} - 1\right) =$$

$$= \frac{1}{n-1}\left(1 - \frac{1}{2^{n-1}}\right)$$

(e) $I = \iint_D \arctan\left(x^2 + \frac{y^2}{2}\right) dx\, dy$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{2} \leq 1,\ x \geq 0, y \geq 0\}$,

Fent $x = r\cos\theta$ i $y = \sqrt{2}\, r \sin\theta$.

$\varphi(r,\theta) = (r\cos\theta, \sqrt{2}\, r\sin\theta)$, $D\varphi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sqrt{2}\sin\theta & \sqrt{2}\, r\cos\theta \end{pmatrix}$, $J_\varphi(r,\theta) = \sqrt{2}\, r$

El domini D^* ve donat per:

$x^2 + \frac{y^2}{2} \leq 1 \Rightarrow (r\cos\theta)^2 + (\sqrt{2}\, r \sin\theta)^2/2 \leq 1 \Rightarrow r^2 \leq 1 \Rightarrow r \in [0,1]$.

Per altra banda, les condicions $x \geq 0$ i $y \geq 0$ ens diuen que som al 1er. quadrant i per tant l'angle $\theta \in [0, \pi/2]$. Així:

$$I = \int_0^{\pi/2} \int_0^1 \sqrt{2}\, r \arctan(r^2)\, dr\, d\theta = \frac{\pi\sqrt{2}}{2} \int_0^1 r \arctan(r^2)\, dr.$$

Així, ens cal calcular:

$$\int r \arctan(r^2)\, dr \underset{\text{canvi: } \{s = r^2,\ ds = 2r\, dr\}}{=} \frac{1}{2}\int \arctan(s)\, ds \underset{\text{Parts}}{=} \frac{1}{2} s \arctan(s) - \frac{1}{2}\int \frac{s}{1+s^2}\, ds =$$

Parts: $\begin{cases} u = \arctan(s) \Rightarrow du = \frac{ds}{1+s^2} \\ dv = ds \Rightarrow v = s \end{cases}$

$$= \frac{s}{2}\arctan(s) - \frac{1}{2}\, \frac{\ln(1+s^2)}{2}$$

Així:

$$I = \frac{\pi\sqrt{2}}{2}\left\{ \frac{1}{2}\arctan(1) - \frac{1}{4}\ln(2) - \left[\frac{0}{2}\arctan(0) - \frac{1}{4}\ln(1)\right] \right\} =$$

$$= \frac{\pi\sqrt{2}}{2}\left(\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\ln(2)\right) = \frac{\pi\sqrt{2}}{8}\left(\frac{\pi}{2} - \ln(2)\right)$$

(g) $I = \iint_D \frac{x+2xy}{x^2+y^2}\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2+1,\ 1 \leq x^2+y^2 \leq e^2,\ x \geq 0\}$

fent $u = x^2+y^2$, $v = y - x^2$.

Atenció: Ara és $\varphi^{-1}(x,y) = (x^2+y^2, y-x^2)$, $D\varphi^{-1}(x,y) = \begin{pmatrix} 2x & 2y \\ -2x & 1 \end{pmatrix}$, $J_{\varphi^{-1}}(x,y) = 2(x+2xy)$

Per tant $J_\varphi(u,v) = \frac{1}{J_{\varphi^{-1}}(x,y)} = \frac{1}{2(x+2y)}$ (que de fet caldria expressar en termes de (u,v)).

El domini D^* ve donat per:

$x^2 \leq y \leq x^2+1 \Rightarrow 0 \leq y - x^2 \leq 1 \Rightarrow 0 \leq -v \leq 1 \Rightarrow -1 \leq v \leq 0.$

$1 \leq x^2+y^2 \leq e^2 \Rightarrow 1 \leq u \leq e^2$

Per tant $D^* = \{(u,v) \in \mathbb{R}^2 : 1 \leq u \leq e^2,\ -1 \leq v \leq 0\}$. Llavors:

$$I = \int_1^{e^2} \left(\int_{-1}^{0} \frac{x+2y}{x^2+y^2} \cdot \frac{1}{2(x+2y)}\, dv \right) du = \frac{1}{2} \int_1^{e^2} \left(\int_{-1}^{0} \frac{dv}{u} \right) du = \frac{1}{2} \int_1^{e^2} \frac{du}{u} = \frac{1}{2} \Big[\ln(u) \Big]_{u=1}^{u=e^2} =$$

$$= \frac{1}{2} (\ln(e^2) - \ln(1)) = 1.$$