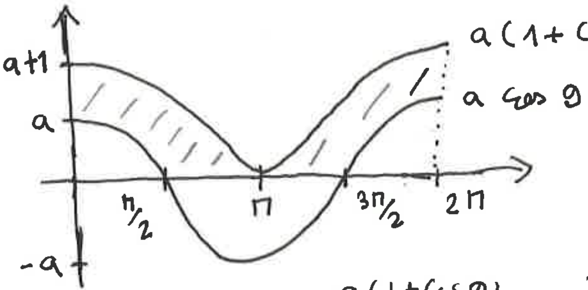
(17) Calculeu les àreas dels dominis $A \subset \mathbb{R}^2$ definits en coordenades polars que s'indiquen tot seguit.

· Observem que si A^* és el domini A en polars, llavors:

$$\text{Àrea}(A) = \iint_A 1\,dx\,dy = \iint_{A^*} r\,dr\,d\theta.$$

(a) A figura definida per $a\cos\theta \le r \le a(1+\cos\theta)$, $(a>0)$.



El valor de r ha de ser dins de la part sombrejada (recordem que sempre $r \ge 0$).

$$\text{Àrea}(A) = \underbrace{\int_0^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr\,d\theta}_{(I)} + \underbrace{\int_{\pi/2}^{3\pi/2}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta}_{(II)} + \underbrace{\int_{3\pi/2}^{2\pi}\int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr\,d\theta}_{(III)}$$

$$(I) = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a\cos\theta}^{r=a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int_0^{\pi/2}(1+2\cos\theta)\,d\theta = \frac{a^2}{2}\left(\frac{\pi}{2}+2\left[\sin\theta\right]_{\theta=0}^{\theta=\pi/2}\right) = \frac{a^2}{2}\left(\frac{\pi}{2}+2\right).$$

$$(II) = \int_{\pi/2}^{3\pi/2}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int_{\pi/2}^{3\pi/2}(1+2\cos\theta+\cos^2\theta)\,d\theta \underset{\uparrow}{=} \frac{a^2}{2}\int_{\pi/2}^{3\pi/2}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{a^2}{2}\left(\frac{3\pi}{2} + 2\left[\sin\theta\right]_{\theta=\pi/2}^{\theta=3\pi/2} + \left[\frac{\sin 2\theta}{4}\right]_{\theta=\pi/2}^{\theta=3\pi/2}\right) =$$

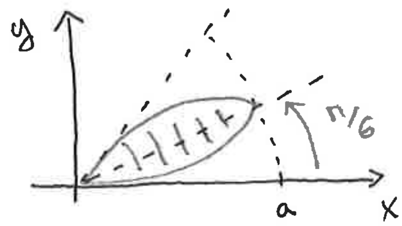
$$= \frac{a^2}{2}\left(\frac{3\pi}{2} + 2\big(\underbrace{\sin\tfrac{3\pi}{2}}_{=-1} - \underbrace{\sin\tfrac{\pi}{2}}_{=1}\big) + \frac{1}{4}\big(\underbrace{\sin 3\pi}_{=0} - \underbrace{\sin\pi}_{=0}\big)\right) = \frac{a^2}{2}\left(\frac{3\pi}{2}-4\right)$$

$(III) = \frac{a^2}{2}\left(\frac{\pi}{2}+2\right)$ [clarament dóna el mateix que (I)]

Així:

$$\text{Àrea}(A) = \frac{a^2}{2}\left(\frac{\pi}{2}+2\right) + \frac{a^2}{2}\left(\frac{3\pi}{2}-4\right) + \frac{a^2}{2}\left(\frac{\pi}{2}+2\right) = \frac{5}{4}\pi a^2.$$

(b) A regió limitada per un pètal de rosa definit per $r = a\sin 3\theta$ $(0 \le \theta \le \pi/3,\ a > 0)$.



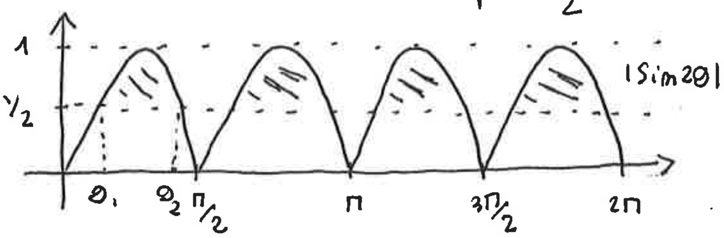
$$\text{Àrea }(A) = \int_0^{\pi/3}\int_0^{a\sin 3\theta} r\,dr\,d\theta = \int_0^{\pi/3}\left[\frac{r^2}{2}\right]_{r=0}^{r=a\sin 3\theta} d\theta =$$

$$= \int_0^{\pi/3} \frac{a^2\sin^2 3\theta}{2}\,d\theta \underset{\uparrow}{=} \frac{a^2}{4}\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta =$$

$$\sin^2 3\theta = \frac{1-\cos 6\theta}{2}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{3} - \left[\frac{\sin 6\theta}{6}\right]_{\theta=0}^{\theta=\pi/3}\right) = \frac{a^2}{4}\left[\frac{\pi}{3} - \left(\frac{\sin(2\pi)}{6} - \frac{\sin 0}{6}\right)\right] = \frac{a^2\pi}{12}$$

(c) A regió definida per $\frac{1}{2} \le r \le |\sin(2\theta)|$.



. A^* és la regió sombrejada

. L'àrea de A és 4 cops la de un dels lòbuls.

θ_1 és t.q. $\sin(2\theta_1) = \frac{1}{2} \Rightarrow 2\theta_1 = \frac{\pi}{6} \Rightarrow \theta_1 = \pi/12$.

θ_2 és t.q. $\sin(2\theta_2) = \frac{1}{2}$ però ara $\theta_2 \in [\pi/4, \pi/2]$. És fàcil veure (i en el dibuix es veu clar!) $\theta_2 = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

$$\text{Àrea }(A) = 4\int_{\theta_1}^{\theta_2}\int_{1/2}^{\sin 2\theta} r\,dr\,d\theta = 4\int_{\theta_1}^{\theta_2}\left[\frac{r^2}{2}\right]_{r=1/2}^{r=\sin 2\theta} d\theta = 2\int_{\theta_1}^{\theta_2}\left(\sin^2 2\theta - \frac{1}{4}\right)d\theta =$$

$$\underset{\uparrow}{=} \int_{\theta_1}^{\theta_2}\left(\frac{1}{2} - \cos 4\theta\right)d\theta = \frac{\theta_2-\theta_1}{2} - \left[\frac{\sin 4\theta}{4}\right]_{\theta=\theta_1}^{\theta=\theta_2} \underset{\uparrow}{=} \frac{\pi}{6} - \frac{1}{4}\left(\sin 4\theta_2 - \sin 4\theta_1\right) =$$

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2} \qquad \theta_2 - \theta_1 = \frac{\pi}{3}$$

$$= \frac{\pi}{6} - \frac{1}{4}\left(\sin(2\pi - \pi/3) - \sin \pi/3\right) = \frac{\pi}{6} + \frac{1}{2}\sin \pi/3 = \frac{\pi}{6} + \frac{\sqrt{3}}{4}.$$

(d) Anàlogament, calculem la integral doble $I = \iint_A \arcsin(x^2+y^2)\,dx\,dy$, on A és la regió limitada per la corba $r = \sqrt{\sin\theta}$, $(0 \le \theta \le \frac{\pi}{2})$.

$$I = \int_0^{\pi/2}\int_0^{\sqrt{\sin\theta}} r\arcsin(r^2)\,dr\,d\theta \underset{\uparrow}{=} \int_0^{\pi/2}\frac{1}{2}\int_0^{\sin\theta}\arcsin(u)\,du\,d\theta$$

canvi: $\{u = r^2,\ du = 2r\,dr\}$

Llavors: $\int \arcsin(u)\,du \underset{\uparrow}{=} u\arcsin(u) - \int \frac{u}{\sqrt{1-u^2}}\,du = u\arcsin(u) + \sqrt{1-u^2}$

Parts: $\left\{\begin{array}{l} U = \arcsin u \Rightarrow dU = \frac{du}{\sqrt{1-u^2}} \\ dV = du \Rightarrow V = u \end{array}\right\}$

d'on:

$$I = \frac{1}{2}\int_0^{\pi/2}\left[\sin\theta\cdot\arcsin(\sin\theta) + \sqrt{1-\sin^2\theta} - \left(0\cdot\arcsin(0) + \sqrt{1-0^2}\right)\right]d\theta = \frac{1}{2}\int_0^{\pi/2}(\theta\sin\theta + \cos\theta - 1)\,d$$

$$\underset{\text{parts}}{=} \frac{1}{2}\left[-\theta\cos\theta + 2\sin\theta - \theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{1}{2}\left[2 - \frac{\pi}{2}\right] = 1 - \pi/4.$$