16. Useu coordenades polars per calcular les següents integrals dobles.

(b) $I = \iint_A \cos(x^2+y^2)\, dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le \frac{\pi}{2}\}$.

coordenades polars: $x = r\cos\theta$, $y = r\sin\theta \iff r^2 = x^2+y^2$, $\theta = \arctan(y/x)$

El domini A en polars és $r^2 \le \pi/2 \iff r \in [0, \sqrt{\pi/2}]$ i $\theta \in [0, 2\pi]$.

$$I = \int_0^{2\pi}\int_0^{\sqrt{\pi/2}} r\cos(r^2)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{\sin(r^2)}{2}\right]_{r=0}^{r=\sqrt{\pi/2}} d\theta = 2\pi\,\frac{\sin(\pi/2)}{2} = \pi.$$

(c) $I = \iint_A \frac{(x+y)^2}{x^2+y^2+2}\, dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$.

$$I = \int_0^{2\pi}\int_0^1 r\,\frac{(r\cos\theta + r\sin\theta)^2}{r^2+2}\, dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,\frac{r^2+2r^2\sin\theta\cos\theta}{r^2+2}\, dr\,d\theta =$$

$$= \int_0^1\int_0^{2\pi} r\,\frac{r^2+r^2\sin 2\theta}{r^2+2}\, d\theta\, dr = \int_0^1 r\,\frac{r^2}{r^2+2}\, dr \cdot \int_0^{2\pi}(1+\sin 2\theta)\, d\theta =$$

$$= \int_0^1 \left(r - \frac{2r}{r^2+2}\right) dr \int_0^{2\pi}(1+\sin 2\theta)\, d\theta = \left[\frac{r^2}{2} - \ln(r^2+2)\right]_{r=0}^{r=1} \cdot \left(2\pi - \left[\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}\right) =$$

$$= \left\{\tfrac{1}{2} - \ln(3) - (0 - \ln(2))\right\} \cdot \left\{2\pi - \left(\tfrac{1}{2} - \tfrac{1}{2}\right)\right\} = 2\pi\left\{\tfrac{1}{2} - \ln(3) + \ln(2)\right\} = 2\pi\left(\tfrac{1}{2} + \ln\left(\tfrac{2}{3}\right)\right)$$

(d) $I = \iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le R^2\}$

$$I = \int_0^{2\pi}\int_0^R \frac{r}{(1+r^2)^2\, r}\, dr\,d\theta = 2\pi\int_0^R \frac{dr}{(1+r^2)^2} \underset{\uparrow}{=} 2\pi\int_0^{\arctan(R)} \frac{1+\tan^2 u}{(1+\tan^2 u)^2}\, du =$$

$r = \tan u$,
$dr = (1+\tan^2 u)\, du$

$$= 2\pi\int_0^{\arctan(R)} \frac{1}{1+\frac{\sin^2 u}{\cos^2 u}}\, du = 2\pi\int_0^{\arctan(R)} \frac{\cos^2 u}{\sin^2 u + \cos^2 u}\, du = 2\pi\int_0^{\arctan(R)} \tfrac{1}{2}(1+\cos 2u)\, du =$$

$$= \pi\left[u + \frac{\sin 2u}{2}\right]_{u=0}^{u=\arctan(R)} = \pi\left[\arctan(R) + \frac{\sin(2\arctan R)}{2}\right]$$

observem: $\frac{\sin(\arctan R)}{\cos(\arctan R)} = \tan(\arctan R) = R$, $\sin^2(\arctan R) + \cos^2(\arctan R) = 1$

d'on: $\sin(\arctan R) = \frac{R}{\sqrt{1+R^2}}$, $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$

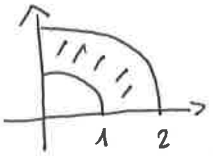
$$I = \pi\left[\arctan(R) + \sin(\arctan R)\cdot\cos(\arctan R)\right] = \pi\left[\arctan R + \frac{R}{1+R^2}\right].$$

(e) $I=\iint_A \sqrt{x^2+y^2-9}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2 : 9\le x^2+y^2\le 25\}$.

El domini A en polars és $r\in[3,5]$, $\theta\in[0,2\pi]$

$$I=\int_0^{2\pi}\int_3^5 r\sqrt{r^2-9}\,dr\,d\theta = 2\pi\left[\frac{(r^2-9)^{3/2}}{\frac{3}{2}\cdot 2}\right]_{r=3}^{r=5} = \frac{2}{3}\pi\left\{16^{3/2}-0\right\}=\frac{128\pi}{3}.$$

(f) $I=\iint_A xy\,dx\,dy$, A intersecció de la corona circular de centre (0,0) i radi interior 1 i exterior 2 amb el 1er. quadrant.



El domini A en polars és $r\in[1,2]$, $\theta\in[0,\pi/2]$.

$$I=\int_0^{\pi/2}\int_1^2 r\,r\cos\theta\, r\sin\theta\,dr\,d\theta = \left(\int_1^2 r^3\,dr\right)\cdot\left(\int_0^{\pi/2}\sin\theta\cdot\cos\theta\,d\theta\right)=$$

$$=\left[\frac{r^4}{4}\right]_{r=1}^{r=2}\cdot\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2}=\frac{1}{4}(2^4-1^4)\cdot\frac{1}{2}(1^2-0^2)=\frac{15}{8}.$$

(g) $I=\iint_A x(x^2+y^2)\,dx\,dy$, A sector circular de centre (0,0) i radi R formant angles entre $\pi/3$ i $\pi/6$ amb l'eix x positiu.

El domini A en polars és $r\in[0,R]$, $\theta\in[\pi/6,\pi/3]$

$$I=\int_0^R\int_{\pi/6}^{\pi/3} r\,r\cos\theta\cdot r^2\,d\theta\,dr=\left(\int_0^R r^4\,dr\right)\cdot\left(\int_{\pi/6}^{\pi/3}\cos\theta\,d\theta\right)=\left[\frac{r^5}{5}\right]_{r=0}^{r=R}\cdot\left[\sin\theta\right]_{\theta=\pi/6}^{\theta=\pi/3}=$$

$$=\frac{R^5}{5}\left(\sin(\pi/3)-\sin(\pi/6)\right)=\frac{R^5}{5}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\frac{\sqrt{3}-1}{10}R^5.$$