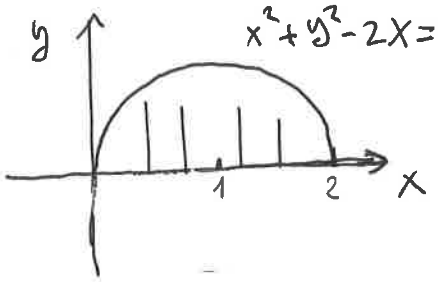
(15) Calculeu les integrals triples següents en les regions que s'indiquen

(a) $I = \iiint_A x z \, dx\, dy\, dz$, A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superfície $z^2=2y$ $(y, z \geq 0)$.

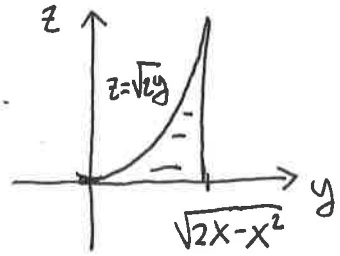
- observem que en el pla (x,y), $x^2+y^2-2x=0 \Leftrightarrow x^2-2x+1+y^2=1$ $\Leftrightarrow (x-1)^2+y^2=1$ és una circumferència de centre $(1,0)$ i radi 1. Si afegim la coordenada z, llavors $x^2+y^2-2x=0$ determina un cilindre en $\mathbb{R}^3$ de base circular.
- La condició $z^2=2y \underset{(z>0)}{\Leftrightarrow} z=\sqrt{2y}$ ens dóna la tapa superior del cilindre, però que només té sentit si $y \geq 0$

base del domini A:

$x^2+y^2-2x=0 \Leftrightarrow y^2=2x-x^2$



secció $x=\text{const.}$ de A:



Parametritzem $A: 0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{2x-x^2},\ 0 \leq z \leq \sqrt{2y}$

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^{\sqrt{2y}} x z \, dz\, dy\, dx = \int_0^2 \int_0^{\sqrt{2x-x^2}} x \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{2y}} dy\, dx =$$

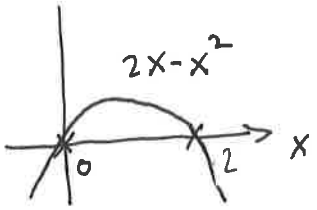
$$= \int_0^2 x \int_0^{\sqrt{2x-x^2}} y\, dy = \int_0^2 x \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{2x-x^2}} dx = \frac{1}{2}\int_0^2 x(2x-x^2)\, dx = \frac{1}{2}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_{x=0}^{x=2} =$$

$$= \frac{1}{2}\left[\frac{2\cdot 8}{3}-\frac{16}{4}\right] = \frac{1}{2}\left[\frac{16-12}{3}\right] = \frac{2}{3}$$

(b) $I=\iiint_A zy\sqrt{x^2+y^2}\,dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^2 : 0\le z\le x^2+y^2,$
$0\le y\le\sqrt{2x-x^2}\}$

· És clar que només ens cal discutir el rang de x.

$2x-x^2=0 \iff x=0$ ó $x=2$.



Per tant, si volem $\sqrt{2x-x^2}$ definida cal

$0\le x\le 2$.

El domini A és: $0\le x\le 2$, $0\le y\le\sqrt{2x-x^2}$, $0\le z\le x^2+y^2$.

$$I=\int_0^2\int_0^{\sqrt{2x-x^2}}\int_0^{x^2+y^2} zy\sqrt{x^2+y^2}\,dz\,dy\,dx=\int_0^2\int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\left[\frac{z^2}{2}\right]_{z=0}^{z=x^2+y^2}dy\,dx=$$

$$=\frac{1}{2}\int_0^2\int_0^{\sqrt{2x-x^2}} y(x^2+y^2)^{5/2}\,dy\,dx=\frac{1}{2}\int_0^2\left[\frac{(x^2+y^2)^{7/2}}{7/2\cdot 2}\right]_{y=0}^{y=\sqrt{2x-x^2}}dx=$$

$$=\frac{1}{14}\int_0^2\left((2x)^{7/2}-x^7\right)dx=\frac{1}{14}\left[2^{7/2}\cdot\frac{x^{9/2}}{9/2}-\frac{x^8}{8}\right]_{x=0}^{x=2}=\frac{1}{14}\left(\frac{2^9}{9}-\frac{2^8}{8}\right)=$$

$$=\frac{2^8}{14}\left(\frac{16-9}{9\cdot 8}\right)=\frac{2^8\cdot 7}{9\cdot 7\cdot 2^4}=\frac{2^4}{9}=\frac{16}{9}.$$

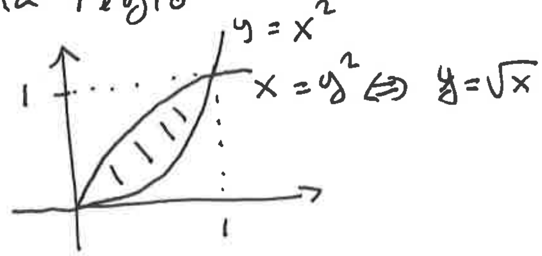
(c) $I=\iiint_A dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^3 : 1\le x\le 3, 1\le y\le 3, 0\le z\le xy\}$

$$I=\int_1^3\int_1^3\int_0^{xy} dz\,dy\,dx=\int_1^3\int_1^3\left[z\right]_{z=0}^{z=xy}dy\,dx=\int_1^3\int_1^3 x\,y\,dy\,dx=$$

$$=\int_1^3 x\,dx\cdot\int_1^3 y\,dy=\left[\frac{x^2}{2}\right]_{x=1}^{x=3}\cdot\left[\frac{y^2}{2}\right]_{y=1}^{y=3}=\left[\frac{1}{2}(3^2-1^2)\right]^2=16$$

(d) $I=\iiint_A xyz\,dx\,dy\,dz$, A limitat per les superfícies $y=x^2$, $x=y^2$, $z=xy$, $z=0$.

- En el pla (x,y), les condicions $y=x^2$, $x=y^2$ determinen la regió



si afegim la variable z, la regió A ve donada per un cilindre on les tapes són $z=0$ i $z=xy$.

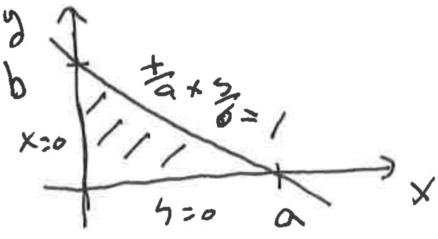
Parametritzem A: $0\le x\le 1$, $x^2\le y\le\sqrt{x}$, $0\le z\le xy$.

$$\underline{I}=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{xy} xyz\,dz\,dy\,dx=\int_0^1\int_{x^2}^{\sqrt{x}} xy\left[\frac{z^2}{2}\right]_{z=0}^{z=xy} dy\,dx=$$

$$=\frac{1}{2}\int_0^1\int_x^{\sqrt{x}} x^3y^3\,dy\,dx=\frac{1}{2}\int_0^1 x^3\left[\frac{y^4}{4}\right]_{y=x}^{y=\sqrt{x}}dx=\frac{1}{8}\int_0^1(x^5-x^7)\,dx=\frac{1}{8}\left(\frac{1}{6}-\frac{1}{8}\right)=\frac{1}{192}$$

(e) $\underline{I}=\iiint_A x\,dx\,dy\,dz$, A tetraedre limitat pels plans $x=0$, $y=0$, $z=0$, $x/a+y/b+z/c=1$ $(a,b,c>0)$.

- És clar que fixats x,y llavors $0\le z\le c(1-x/a-y/b)$.
- En el pla $z=0$ el tetraedre determina el triangle de costats: $x=0$, $y=0$ i $x/a+y/b=1$



Per tant A és el domini: $0\le x\le a$, $0\le y\le b\left(1-\frac{x}{a}\right)$, $0\le z\le c\left(1-x/a-y/b\right)$.

$$I=\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} x\,dz\,dy\,dx=\int_0^a\int_0^{b(1-x/a)} cx\,(1-x/a-y/b)\,dy\,dx=$$

$$=c\int_0^a x\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_{y=0}^{y=b(1-x/a)}dx=c\int_0^a x\left\{\left(1-\frac{x}{a}\right)b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right\}dx=$$

$$=\frac{bc}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2dx=\frac{bc}{2}\int_0^a x\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)dx=\frac{bc}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3a}+\frac{x^4}{4a^2}\right]_{x=0}^{x=a}=\frac{a^2bc}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{a^2bc}{24}$$