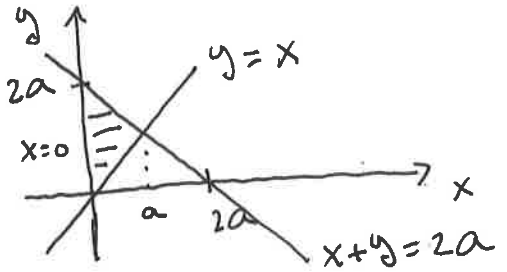
(12) Calculen les següents integrals dobles en els dominis que s'indiquen.

(a) $I = \iint_A (x^2+y^2)\,dx\,dy$, A limitat per les rectes $y = x$, $x+y = 2a$, $x = 0$ $(a>0)$.



Parametritzem A:

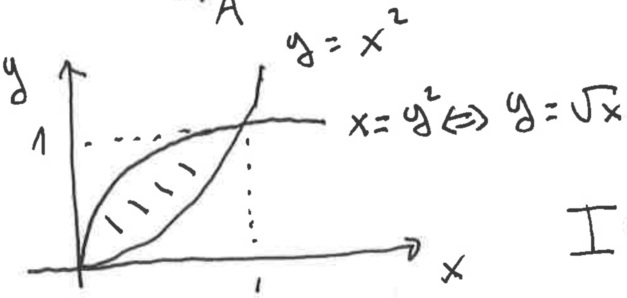
$$\{0 \le x \le a,\ x \le y \le 2a-x\}$$

$$I = \int_0^a \left(\int_x^{2a-x} (x^2+y^2)\,dy \right) dx =$$

$$= \int_0^a \left[x^2 y + y^3/3\right]_{y=x}^{y=2a-x} dx = \int_0^a \left[x^2(2a-x) + \frac{(2a-x)^3}{3} - \left(x^3 + \frac{x^3}{3}\right)\right] dx =$$

$$= \left[2a\frac{x^3}{3} - \frac{x^4}{4} - \frac{(2a-x)^4}{12} - \frac{x^4}{3}\right]_{x=0}^{x=a} = 2\frac{a^4}{3} - \frac{a^4}{4} - \frac{a^4}{12} - \frac{a^4}{3} - \left(-\frac{(2a)^4}{12}\right) = \frac{4}{3}a^4.$$

(b) $I = \iint_A (x+2y)\,dx\,dy$, A limitat per los corbos $y = x^2$, $y^2 = x$



Parametritzem A:

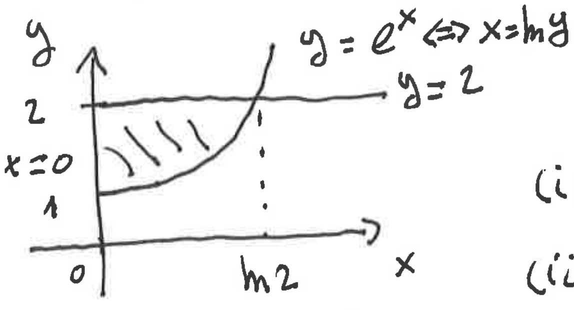
$$\{0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$$

$$I = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (x+2y)\,dy \right) dx =$$

$$= \int_0^1 \left[xy + y^2\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left[x\sqrt{x} + x - (x^3 + x^4)\right] dx =$$

$$= \left[\frac{x^{5/2}}{5/2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^5}{5}\right]_{x=0}^{x=1} = \frac{2}{5} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{9}{20}.$$

(c) $I = \iint_A e^{x+y}\,dx\,dy$, A limitat per les corbes $y = e^x$, $x = 0$, $y = 2$



Parametritzem A: D'entrada hi ha 2 possibilitats

(i) $0 \le x \le \ln 2$, $e^x \le y \le 2$

(ii) $1 \le y \le 2$, $0 \le x \le \ln y$

Que donen:

$$(i)\ I = \int_0^{\ln 2}\left(\int_{e^x}^{2} e^{x+y}\,dy\right)dx\ ;\quad (ii)\ I = \int_1^2\left(\int_0^{\ln y} e^{x+y}\,dx\right)dy$$

Fent càlculs vegem que (ii) és abordable, però (i) no!

$$(ii)\ I = \int_1^2 \left[e^{x+y}\right]_{x=0}^{x=\ln y} dy = \int_1^2 (e^{\ln y + y} - e^{y})\,dy = \int_1^2 (y e^y - e^y)\,dy =$$

$$\underset{\uparrow}{=} \left[y e^y - 2e^y\right]_{y=1}^{y=2} = 2e^2 - 2e^2 - (e - 2e) = e.$$

$\left(\int y e^y\,dy = y e^y - e^y \text{ (parts)}\right)$