(9) Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen.

(a) $I = \iint_A y^3\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \le x \le \frac{\pi}{2}, 0 \le y \le 2\cos x\}$

$$I = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2\cos x} y^3\,dy \right) dx = \int_{-\pi/2}^{\pi/2} \left[\frac{y^4}{4} \right]_{y=0}^{y=2\cos x} dx = \int_{-\pi/2}^{\pi/2} 4\cos^4 x\,dx =$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^2 x\,(1-\sin^2 x)\,dx \underset{\uparrow}{=} \int_{-\pi/2}^{\pi/2} 4\left(\cos^2 x - \frac{1}{4}\sin^2(2x)\right) dx \underset{\uparrow}{=}$$

$\sin 2x = 2\sin x \cos x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2(2x) = \frac{1-\cos 4x}{2}$

$$= \int_{-\pi/2}^{\pi/2} 2(1+\cos 2x)\,dx - \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1-\cos(4x))\,dx =$$

$$= \frac{3}{2}\int_{-\pi/2}^{\pi/2} dx + \int_{-\pi/2}^{\pi/2} \left(2\cos 2x + \frac{1}{2}\cos 4x\right) dx = \frac{3}{2}\pi + \left[\sin 2x + \frac{\sin 4x}{8}\right]_{x=-\pi/2}^{x=\pi/2} =$$

$= \frac{3}{2}\pi$ (ja que $\sin(\pm\pi) = \sin(\pm 2\pi) = 0$).

(d) $I = \iint_A \frac{x^2}{y^2}\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2, \frac{1}{x} \le y \le x\}$

$$I = \int_1^{2} \left(\int_{1/x}^{x} \frac{x^2}{y^2}\,dy \right) dx = \int_1^2 x^2 \left(\int_{1/x}^{x} y^{-2}\,dy \right) dx = \int_1^2 x^2 \left[\frac{y^{-1}}{-1} \right]_{y=1/x}^{y=x} dx =$$

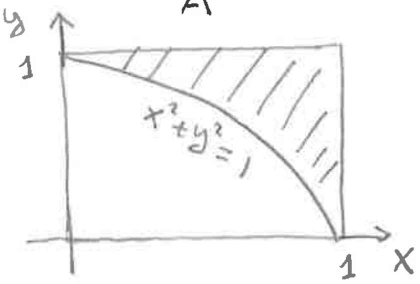
$$= \int_1^2 x^2 \left(x - \frac{1}{x}\right) dx = \int_1^2 (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{x=1}^{x=2} = \frac{16}{4} - \frac{4}{2} - \left(\frac{1}{4} - \frac{1}{2}\right) =$$

$$= \frac{16-8-1+2}{4} = \frac{9}{4}.$$

(f) $I=\iint_A y^3\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2: 0\le x\le a,\ y^2\le 2px\}$, $a,p>0$.

$$I=\int_0^a\left(\int_{-\sqrt{2px}}^{\sqrt{2px}} y^3\,dy\right)dx=\int_0^a\left[\frac{y^4}{4}\right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}}dx=\int_0^a p^2[x^2-x^2)\,dx=0.$$

(g) $I=\iint_A \frac{y}{1+x^3}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2: 0\le x\le 1,\ 0\le y\le 1,\ x^2+y^2\ge 1\}$



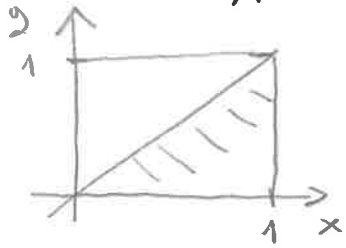
Com que la funció que hem d'integrar és més fàcil fer-ho respecte y que respecte x, parametritzem el domini A respecte x.

$x\in[0,1]\Rightarrow\sqrt{1-x^2}\le y\le 1$

$$I=\int_0^1\left(\int_{\sqrt{1-x^2}}^1\frac{y}{1+x^3}\,dy\right)dx=\int_0^1\frac{1}{1+x^3}\left[\frac{y^2}{2}\right]_{y=\sqrt{1-x^2}}^{y=1}dx=\frac{1}{2}\int_0^1\frac{1}{1+x^3}[1-(1-x^2)]\,dx=$$

$$=\frac{1}{2}\int_0^1\frac{x^2}{1+x^3}\,dx=\frac{1}{2}\left[\frac{\ln(1+x^3)}{3}\right]_{x=0}^{x=1}=\frac{1}{6}(\ln(2)-\ln(1))=\frac{\ln(2)}{6}.$$

(h) $I=\iint_A x^2\sin(xy)\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2: 0\le y\le 1,\ y\le x\le 1\}$.



La parametrització que ens donen del domini A és resp. y, però és molt més fàcil integrar la funció resp. x. Per tant reparametritzem:

$x\in[0,1]\Rightarrow 0\le y\le x$

$$I=\int_0^1\left(\int_0^x x^2\sin(xy)\,dy\right)dx=\int_0^1 x^2\left[-\frac{\cos(xy)}{x}\right]_{y=0}^{y=x}dx=$$

$$=\int_0^1(x-x\cos(x^2))\,dx=\left[\frac{x^2}{2}-\frac{\sin(x^2)}{2}\right]_{x=0}^{x=1}=\frac{1-\sin(1)}{2}.$$