(47) Calculeu el desenvolupament de Taylor en l'origen de les següents funcions fins l'ordre que s'indica en cada cas i doneu el valor de totes les derivades parcials de la funció en el (0,0), corresponents a l'ordre màxim fins al qual s'ha desenvolupat. (P. ex., si desenvolupem fins ordre 5 volem $\frac{\partial^5 f}{\partial x^n y^m}(0,0)$ amb $n+m=5$.)

Recordem: El coeficient que multiplica $x^n y^m$ en el desenvolupament de Taylor de $f(x,y)$ en $(0,0)$ és

$$\frac{1}{n!\,m!}\cdot\frac{\partial^{n+m} f}{\partial x^n \partial y^m}(0,0)$$

(a) $f(x,y) = \ln(1+x^2-y)$ fins ordre 3.

$$f(x,y) = x^2 - y - \frac{(x^2-y)^2}{2} + \frac{(x^2-y)^3}{3} + O_4(x,y) =$$

$$= x^2 - y - \frac{(x^4 - 2yx^2 + y^2)}{2} + \frac{x^6 - 3x^4y + 3x^2y^2 - y^3}{3} + O_4(x,y)$$

$$= -y + x^2 - y^2/2 + x^2y - y^3/3 + O_4(x,y).$$

clarament:

$$\frac{\partial^3 f}{\partial x^3}(0,0) = \frac{\partial^3 f}{\partial x \partial y^2}(0,0) = 0.$$

A més: (Recordem: $0! = 1$)

$$\frac{1}{2!\,1!}\frac{\partial^3 f}{\partial x^2 \partial y}(0,0) = 1 \Rightarrow \frac{\partial^3 f}{\partial x^2 \partial y}(0,0) = 2! = 2.$$

$$\frac{1}{0!\,3!}\frac{\partial^3 f}{\partial y^3}(0,0) = -1/3 \Rightarrow \frac{\partial^3 f}{\partial y^3}(0,0) = -\frac{3!}{3} = -2.$$

(b) $f(x,y) = \cos(xy)$ fins ordre 8.

$$f(x,y) = 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} + o_{10}(x,y) = 1 - \frac{x^2y^2}{2} + \frac{x^4y^4}{24} + o_{10}(x,y)$$

claramente totes les derivades d'ordre 8 en $(0,0)$ són nules tret de:

$$\frac{1}{4!\,4!}\frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{1}{24} \Rightarrow \frac{\partial^8 f}{\partial x^4 y^4}(0,0) = \frac{4!\,4!}{24} = 24$$

(c) $f(x,y) = e^{x^2-y^2}$ fins ordre 8.

$$f(x,y) = e^{x^2-y^2} = 1 + (x^2-y^2) + \frac{(x^2-y^2)^2}{2!} + \frac{(x^2-y^2)^3}{3!} + \frac{(x^2-y^2)^4}{4!} + o_{10}(x,y)$$

$$= 1 + x^2 - y^2 + \frac{1}{2}(x^4 - 2x^2y^2 + y^4) +$$

$$+ \frac{1}{6}(x^6 - 3x^4y^2 + 3x^2y^4 - y^6) +$$

$$+ \frac{1}{24}(x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8) + o_{10}(x,y)$$

Les derivades d'ordre 8 no nules són:

$$\frac{1}{8!\,0!}\frac{\partial^8 f}{\partial x^8}(0,0) = \frac{1}{24} \Rightarrow \frac{\partial^8 f}{\partial x^8}(0,0) = \frac{8!}{24} = 1680$$

$$\frac{1}{6!\,2!}\frac{\partial^8 f}{\partial x^6 \partial y^2}(0,0) = -\frac{4}{24} \Rightarrow \frac{\partial^8 f}{\partial x^6 \partial y^2}(0,0) = -\frac{4\cdot 6!\,2!}{24} = -240$$

$$\frac{1}{4!\,4!}\frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{6}{24} \Rightarrow \frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{6\cdot 4!\,4!}{24} = 144$$

$$\frac{1}{2!\,6!}\frac{\partial^8 f}{\partial x^2 \partial y^6}(0,0) = \frac{-4}{24} \Rightarrow \frac{\partial^8 f}{\partial x^2 \partial y^6}(0,0) = -240.$$

$$\frac{1}{0!\,8!}\frac{\partial^8 f}{\partial y^8}(0,0) = \frac{1}{24} \Rightarrow \frac{\partial^8 f}{\partial y^8}(0,0) = 1680$$