(28) Calculeu totes les derivades parcials fins ordre 2 de les següents funcions i doneu el seu desenvolupament de taylor fins a termes de grau 2 inclosos entorn del punt que s'indica en cada cas.

(a) taylor de $f(x,y) = \sin(xy)$, entorn del punt $(1, \pi/2)$.

$$\frac{\partial f}{\partial x} = \cos(xy)\cdot y \ ; \quad \frac{\partial f}{\partial y} = \cos(xy)\cdot x \ ; \quad \frac{\partial^2 f}{\partial x^2} = -\sin(xy)\, y^2 ;$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(xy)\, xy + \cos(xy) \ ; \quad \frac{\partial^2 f}{\partial y^2} = -\sin(xy)\, x^2 .$$

Així: $f(1, \pi/2) = 1$, $\frac{\partial f}{\partial x}(1, \pi/2) = 0$, $\frac{\partial f}{\partial y}(1, \pi/2) = 0$,

$\frac{\partial^2 f}{\partial x^2}(1, \pi/2) = -\pi^2/4$, $\frac{\partial^2 f}{\partial x \partial y}(1, \pi/2) = -\pi/2$, $\frac{\partial^2 f}{\partial y^2}(1, \pi/2) = -1$.

Per tant:

$$f(x,y) = f(1, \pi/2) + \frac{\partial f}{\partial x}(1, \pi/2)\cdot(x-1) + \frac{\partial f}{\partial y}(1, \pi/2)\cdot(y - \pi/2) +$$

$$+ \frac{1}{2!}\left( \frac{\partial^2 f}{\partial x^2}(1, \pi/2)\cdot(x-1)^2 + 2\cdot \frac{\partial^2 f}{\partial x \partial y}(1, \pi/2)\,(x-1)\cdot(y - \pi/2) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial y^2}(1, \pi/2)\cdot(y - \pi/2)^2 \right) + R_2(x,y) =$$

$$= 1 + 0\cdot(x-1) + 0\cdot(y - \pi/2) +$$

$$+ \frac{1}{2}\left( -\frac{\pi^2}{4}(x-1)^2 + 2\cdot\left(-\frac{\pi}{2}\right)\cdot(x-1)(y - \tfrac{\pi}{2}) + (-1)(y - \tfrac{\pi}{2})^2 \right) + R_2(x,y)$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y - \tfrac{\pi}{2}) - \frac{1}{2}(y - \tfrac{\pi}{2})^2 + R_2(x,y).$$

(b) taylor de $f(x,y) = x^y$ entorn del punt $(1,1)$.

$f = e^{y\ln x}$ ; $\frac{\partial f}{\partial x} = e^{y \ln x}\frac{y}{x} = x^y \frac{y}{x}$; $\frac{\partial f}{\partial y} = x^y \ln x$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(x^y \frac{y}{x}\right) = x^y \frac{y^2}{x^2} + x^y\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(x^y \frac{y}{x}\right) = x^y \ln x \, \frac{y}{x} + x^y \frac{1}{x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x^y \ln x) = x^y (\ln x)^2$$

$f(1,1) = 1$, $\frac{\partial f}{\partial x}(1,1) = 1$, $\frac{\partial f}{\partial y}(1,1) = 0$,

$\frac{\partial^2 f}{\partial x^2}(1,1) = 0$, $\frac{\partial^2 f}{\partial x \partial y}(1,1) = 1$, $\frac{\partial^2 f}{\partial y^2}(1,1) = 0$.

$$f(x,y) = 1 + 1\cdot(x-1) + 0\cdot(y-1) + \frac{1}{2}\left\{0\cdot(x-1)^2 + 2\cdot 1\cdot(x-1)(y-1) + 0\cdot(y-1)^2\right\} + R_2(x,y) = 1 + (x-1) + (x-1)(y-1) + R_2(x,y)$$

(c) taylor de $f(x,y) = e^{x/y}$ entorn del punt $(0,1)$.

$\frac{\partial f}{\partial x} = e^{x/y}\frac{1}{y}$ ; $\frac{\partial f}{\partial y} = -e^{x/y}\frac{x}{y^2}$, $\frac{\partial^2 f}{\partial x^2} = e^{x/y}\frac{1}{y^2}$,

$\frac{\partial^2 f}{\partial x \partial y} = e^{x/y}\left(-\frac{x}{y^3} - \frac{1}{y^2}\right)$, $\frac{\partial^2 f}{\partial y^2} = e^{x/y}\left(\frac{x}{y^4} + \frac{2x}{y^3}\right)$.

$f(0,1) = 1$, $\frac{\partial f}{\partial x}(0,1) = 1$, $\frac{\partial f}{\partial y}(0,1) = 0$, $\frac{\partial^2 f}{\partial x^2}(0,1) = 1$,

$\frac{\partial^2 f}{\partial x \partial y}(0,1) = -1$, $\frac{\partial^2 f}{\partial y^2}(0,1) = 0$

$$f(x,y) = 1 + x + \frac{1}{2}x^2 - x(y-1) + R_2(x,y)$$

(d) Taylor de $f(x,y,z) = e^{-x}\sin(yz)$ entorn del punt $(0,1,\pi)$.

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$$\frac{\partial f}{\partial x} = -e^{-x}\sin(yz), \quad \frac{\partial f}{\partial y} = e^{-x}\cos(yz)\,z, \quad \frac{\partial f}{\partial z} = e^{-x}\cos(yz)\cdot y,$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-x}\sin(yz), \quad \frac{\partial^2 f}{\partial x \partial y} = -e^{-x}\cos(yz)\,z, \quad \frac{\partial^2 f}{\partial x \partial z} = -e^{-x}\cos(yz)\,y,$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{-x}\sin(yz)\,z^2, \quad \frac{\partial^2 f}{\partial y \partial z} = e^{-x}\{-\sin(yz)\,y\cdot z + \cos(yz)\},$$

$$\frac{\partial^2 f}{\partial z^2} = -e^{-x}\sin(yz)\,y^2$$

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$$f(0,1,\pi) = e^{-0}\cdot\sin(1\cdot\pi) = \sin(\pi) = 0, \quad \frac{\partial f}{\partial x}(0,1,\pi) = 0,$$

$$\frac{\partial f}{\partial y}(0,1,\pi) = \cos(\pi)\cdot\pi = -\pi, \quad \frac{\partial f}{\partial z}(0,1,\pi) = \cos(\pi)\cdot 1 = -1,$$

$$\frac{\partial^2 f}{\partial x^2}(0,1,\pi) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0,1,\pi) = \pi, \quad \frac{\partial^2 f}{\partial x \partial z}(0,1,\pi) = 1,$$

$$\frac{\partial^2 f}{\partial y^2}(0,1,\pi) = 0, \quad \frac{\partial^2 f}{\partial y \partial z}(0,1,\pi) = -1, \quad \frac{\partial^2 f}{\partial z^2}(0,1,\pi) = 0.$$

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$$f(x,y,z) = f(0,1,\pi) + \frac{\partial f}{\partial x}(0,1,\pi)\cdot(x-0) + \frac{\partial f}{\partial y}(0,1,\pi)\cdot(y-1) +$$

$$+ \frac{\partial f}{\partial z}(0,1,\pi)\cdot(z-\pi) + \frac{1}{2}\Big\{\frac{\partial^2 f}{\partial x^2}(0,1,\pi)(x-0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,1,\pi)\cdot(x-0)(y-1) +$$

$$+ 2\frac{\partial^2 f}{\partial x \partial z}(0,1,\pi)\cdot(x-0)(z-\pi) + \frac{\partial^2 f}{\partial y^2}(0,1,\pi)\cdot(y-1)^2 + 2\frac{\partial^2 f}{\partial y \partial z}(0,1,\pi)\cdot(y-1)(z-\pi) +$$

$$+ \frac{\partial^2 f}{\partial z^2}(0,1,\pi)\cdot(z-\pi)^2\Big\} + R_2(x,y,z) =$$

$$= -\pi\cdot(y-1) - (z-\pi) + \pi\cdot\cancel{x}\cdot(y-1) + x(z-\pi) - (y-1)(z-\pi) + R_2(x,y,z)$$