(32) Direm que $f(x,y)$ és homogènia de grau $m \in \mathbb{N}$ si:

$$f(t\cdot x,\ t\cdot y) = t^m \cdot f(x,y),\quad \forall t \in \mathbb{R}$$

(a) Demostreu que si $f \in C^1$, llavors $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = m f(x,y)$.

Derivem la identitat de l'enunciat respecte de t:

$$\frac{\partial f}{\partial x}(tx,ty)\frac{\partial}{\partial t}(tx) + \frac{\partial f}{\partial y}(tx,ty)\frac{\partial}{\partial t}(ty) = m t^{m-1} f(x,y)$$

Ara fem $t=1$: $\frac{\partial f}{\partial x}(x,y)\cdot x + \frac{\partial f}{\partial y}(x,y)\, y = m f(x,y)$

(b) Si $f \in C^2$, $m=1$ i $(x,y) \neq (0,0)$, vegeu que $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}$.

Derivem el resultat de (a) respecte x i y per $m=1$:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial f}{\partial x} \\ x\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} &= \frac{\partial f}{\partial y} \end{aligned}\right\}\quad \left.\begin{aligned} x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} &= 0 \\ x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} &= 0 \end{aligned}\right\}$$

D'on:

$$\underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}}_{\substack{\parallel \\ A}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Sabem doncs que (x,y) és solució del sistema d'equacions lineal i homogeni definit per la matriu A. Llavors com que $(x,y) \neq (0,0)$ hem de tenir $\det A = 0$. Això dóna la condició que busquem.