(11) Calculeu les derivades parcials ~~segones~~ segones de les següents funcions i doneu el seu hessià. (hessià = matriu derivades 2ones).

(a) $f(x,y) = \sin x \cdot \sin^2 y$.

$\frac{\partial f}{\partial x} = \cos x \sin^2 y$, $\frac{\partial f}{\partial y} = 2 \sin x \sin y \cos y$, $\frac{\partial^2 f}{\partial x^2} = -\sin x \cdot \sin^2 y$,

$\frac{\partial^2 f}{\partial x \partial y} = \underbrace{2 \cos x \sin y \cos y}_{= \cos x \sin 2y}$, $\frac{\partial^2 f}{\partial y^2} = 2 \sin x (\cos^2 y - \sin^2 y) = 2 \sin x \cdot \cos 2y$

$$H_f = \begin{bmatrix} -\sin x \cdot \sin^2 y & \cos x \cdot \sin 2y \\ \cos x \cdot \sin 2y & 2 \sin x \cos 2y \end{bmatrix}$$

(b) $f(x,y) = \sin(x^2 - 3xy)$

$\frac{\partial f}{\partial x} = \cos(x^2 - 3xy) \cdot (2x - 3y)$, $\frac{\partial f}{\partial y} = \cos(x^2 - 3xy) \cdot (-3x)$,

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$\frac{\partial^2 f}{\partial x^2} = -\sin(x^2 - 3xy)(2x - 3xy)^2 + 2 \cos(x^2 - 3xy)$

$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x^2 - 3xy)(-3x)(2x - 3y) + \cos(x^2 - 3xy)(-3)$

$\frac{\partial^2 f}{\partial y^2} = -\sin(x^2 - 3xy)(-3x)^2$

(c) $f(x,y) = x \arctan\left(\frac{x}{y}\right)$.

$\frac{\partial f}{\partial x} = \arctan\left(\frac{x}{y}\right) + x \frac{1/y}{1 + (x/y)^2} = \arctan\left(\frac{x}{y}\right) + \frac{xy}{x^2 + y^2}$

$\frac{\partial f}{\partial y} = x \frac{-x/y^2}{1 + (x/y)^2} = -\frac{x^2}{x^2 + y^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{1/y}{1 + (x/y)^2} + \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y}{x^2 + y^2} + \frac{y^3 - x^2 y}{(x^2 + y^2)^2}$

$\frac{\partial^2 f}{\partial \cancel{x} \partial y} = -\frac{2x(x^2 + y^2) - x^2(2x)}{(x^2 + y^2)^2} = \frac{-2xy^2}{(x^2 + y^2)^2}$

$\frac{\partial^2 f}{\partial y^2} = + \frac{x^2\, 2y}{(x^2 + y^2)^2}$

(d) $f(x,y) = \exp\left(-\frac{1}{x^2+y^2}\right)$

$$\frac{\partial f}{\partial x} = \exp\left(-\frac{1}{x^2+y^2}\right) \cdot \frac{2x}{(x^2+y^2)^2} = \frac{2x}{(x^2+y^2)^2} f = 2x \cdot (x^2+y^2)^{-2} f$$

$$\frac{\partial f}{\partial y} = \exp\left(-\frac{1}{x^2+y^2}\right) \cdot \frac{2y}{(x^2+y^2)^2} = \frac{2y}{(x^2+y^2)^2} f$$

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{2}{(x^2+y^2)^2} + \frac{2x\cdot(-2)\cdot 2x}{(x^2+y^2)^3}\right) f + \frac{2x}{(x^2+y^2)^2}\underbrace{\frac{2x}{(x^2+y^2)^2} f}_{\frac{\partial f}{\partial x}} =$$

$$= \frac{2f}{(x^2+y^2)^4}\Big[\underbrace{(x^2+y^2)^2 - 4x^2(x^2+y^2) + 2x^2}_{= x^4 + 2x^2y^2 + y^4 - 4x^4 - 4x^2y^2 + 2x^2}\Big] =$$

$$= \frac{2f}{(x^2+y^2)^4}\left[-3x^4 - 2x^2y^2 + y^4 + 2x^2\right]$$

Per simetria: $\frac{\partial^2 f}{\partial y^2} = \frac{2f}{(x^2+y^2)^4}\left[-3y^4 - 2x^2y^2 + x^4 + 2y^2\right]$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x(-2)(x^2+y^2)^{-3}(2y) f + 2x(x^2+y^2)^{-2}\left(\frac{2y}{(x^2+y^2)^2} f\right) =$$

derivem $\frac{\partial f}{\partial x}$ resp. y

$$= \frac{4fxy}{(x^2+y^2)^4}\left[-2(x^2+y^2) + 1\right]$$

(e) $f(x,y,z) = xy^2z^2e^x$

$\frac{\partial f}{\partial x} = y^2z^2(1+x)e^x$; $\frac{\partial f}{\partial y} = 2xyz^2e^x$; $\frac{\partial f}{\partial z} = 2xy^2ze^x$, $H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$

$\frac{\partial^2 f}{\partial x^2} = y^2z^2(2+x)e^x$; $\frac{\partial^2 f}{\partial x \partial y} = 2yz^2(1+x)e^x$; $\frac{\partial^2 f}{\partial x \partial z} = 2y^2z(1+x)e^x$,

$\frac{\partial^2 f}{\partial y^2} = 2xz^2e^x$; $\frac{\partial^2 f}{\partial y \partial z} = 4xyze^x$; $\frac{\partial^2 f}{\partial z^2} = 2xy^2e^x$

(f) $f(x,y,z) = x^2y + xy^2 + yz^2$

$\frac{\partial f}{\partial x} = 2xy + y^2$; $\frac{\partial f}{\partial y} = x^2 + 2xy + z^2$; $\frac{\partial f}{\partial z} = 2yz$,

$\frac{\partial^2 f}{\partial x^2} = 2y$; $\frac{\partial^2 f}{\partial x \partial y} = 2x + 2y$; $\frac{\partial^2 f}{\partial x \partial z} = 0$; $\frac{\partial^2 f}{\partial y^2} = 2x$; $\frac{\partial^2 f}{\partial y \partial z} = 2z$,

$\frac{\partial^2 f}{\partial z^2} = 2y$