29 (a) Trobeu la sèrie de Fourier de $f(x) = x + \pi$ en l'interval $-\pi < x < \pi$.

- Fem $f(x) = \pi + g(x)$, on $g(x) = x$ és una funció senar. Per tant, el desenvolupament de Taylor de $g(x)$ només té sinus.

$$g(x) \sim \mathcal{F}[g](x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{T} x\right) + b_n \sin\left(\frac{n\pi}{T} x\right)\right)$$

on $T = \pi$ i $a_n = 0$, $\forall n = 0, 1, 2, \dots$

$$b_n = \frac{1}{T} \int_{-T}^{T} \underbrace{g(x) \sin\left(\frac{n\pi}{T} x\right)}_{} = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx =$$

g(x) senar: podem integrar només en mig interval

$$= \left\{ \begin{matrix} u = x \rightarrow du = dx \\ dv = \sin(nx)\,dx \rightarrow v = -\frac{1}{n}\cos(nx) \end{matrix} \right\} = \frac{2}{\pi} \left\{ \left[-\frac{x}{n}\cos(nx) \right]_{x=0}^{x=\pi} + \frac{1}{n}\int_0^{\pi} \cos(nx)\,dx \right\} =$$

$$= \frac{2}{\pi} \left\{ \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\left[\frac{\sin(nx)}{n}\right]_{x=0}^{x=\pi} \right\} = \frac{2(-1)^{n+1}}{n}$$

$\cos(n\pi) = (-1)^n$

$$f(x) = \pi + g(x) \sim \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) = \mathcal{F}[f](x)$$

(b) Useu (a) per demostrar $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

T^a Dirichlet: f i f' són contínues a trossos en $[-\pi, \pi]$.

Llavors, $\mathcal{F}(x_0) = \frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right)$, on:

$$f(x_0^+) = \lim_{x\to x_0^+} f(x), \quad f(x_0^-) = \lim_{x \to x_0^-} f(x).$$

Si tem $x_0 = \pi/2$ i usem que f és contínua en $\pi/2$

i que per tant: $f(\pi/2^+) = f(\pi/2^-) = f(\pi/2) = 3\pi/2$,

llavors:

$$3\frac{\pi}{2} = f(\pi/2) = \mathcal{F}[f](\pi/2) = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(n\frac{\pi}{2}\right) =$$

$$= \pi + \sum_{k=1}^{\infty} \frac{2(-1)^{2k+1}}{(2k)} \sin(k\pi) + \sum_{k=1}^{\infty} \frac{2(-1)^{2k-1+1}}{(2k-1)} \sin\left((2k-1)\frac{\pi}{2}\right) =$$

trenquem la suma en $\underbrace{\text{parells}}_{n=2k}$ i $\underbrace{\text{senars}}_{n=2k-1}$

$$= \pi + \sum_{k=1}^{\infty} \frac{2}{(2k-1)} (-1)^{k+1} \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}}$$

$\sin(k\pi) = 0$, $\sin\left((2k-1)\frac{\pi}{2}\right) = (-1)^{k+1} = \begin{cases} 1, & k \text{ parell} \\ -1, & k \text{ senar} \end{cases}$