11) Troben l'interval de convergència de les sèries de potències:

(a) $\displaystyle\sum_{n=0}^{\infty} \underbrace{\left(\frac{x}{4}\right)^n}_{u_n}$

Criteri de l'arrel: $\displaystyle\lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \left|\frac{x}{4}\right| = \frac{|x|}{4} < 1 \iff |x| < 4$

Radi de convergència $R = 4 \Rightarrow$ convergeix absolutament en $(-4, 4)$.

$x = 4$: $\displaystyle\sum_{n=0}^{\infty} 1$ divergent ; $x = -4$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ divergent

Interval de convergència : $x \in (-4, 4)$.

(b) $\displaystyle\sum_{n=0}^{\infty} \underbrace{(2n)!\left(\frac{x}{3}\right)^n}_{u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+2)!\left(\frac{x}{3}\right)^{n+1}}{(2n)!\left(\frac{x}{3}\right)^n}\right| =$

$\displaystyle= \lim_{n\to\infty} (2n+2)(2n+1)\frac{|x|}{3} = +\infty > 1$, si $x \neq 0$.

Radi de convergència $R = 0$. Només convergeix si $x = 0$.

(c) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(x-3)^{n-1}}{3^n}}_{u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-3)^n}{3^{n+1}}}{\frac{(x-3)^{n-1}}{3^n}}\right| =$

$\displaystyle= \lim_{n\to\infty} \frac{|x-3|}{3} = \frac{|x-3|}{3} < 1 \iff |x-3| < 3$

Radi de conv. $R = 3 \Rightarrow$ conv. abs. en $(3-3, 3+3) = (0, 6)$.

$x = 0$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{3}$ div. ; $x = 6$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3}$ div.

Interval de conv. : $x \in (0, 6)$.

(d) $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n x^{2n}}{n!}}_{u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| = \lim_{n\to\infty}\frac{x^2}{n+1} = 0 < 1,\ \forall x\in\mathbb{R}$

Radi de conv. $R = +\infty \Rightarrow$ Convergeix $\forall x \in \mathbb{R}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{n!\, x^{2n}}{(2n)!}}_{u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!\, x^{2n+2}}{(2n+2)!}}{\frac{n!\, x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\frac{(n+1)\, x^2}{(2n+2)(2n+1)} =$

$= 0 < 1,\ \forall x\in\mathbb{R}$. Radi conv. $R = +\infty \Rightarrow$ conv. $\forall x\in\mathbb{R}$.

(f) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot 3\cdots(n+1)\, x^n}{n!} = \sum_{n=1}^{\infty}\underbrace{(n+1)\, x^n}_{u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\, x^{n+1}}{(n+1)\, x}\right| = |x| < 1 \Leftrightarrow |x| < 1$

Radi conv. $R = 1 \Rightarrow$ conv. abs. $\forall x \in (-1,1)$.

$x = 1$: $\displaystyle\sum_{n=1}^{\infty}(n+1)$ div. ; $x = -1$: $\displaystyle\sum_{n=1}^{\infty}(n+1)(-1)^n$ div.

Interval de conv. $x\in(-1,1)$.

(g) $\displaystyle\sum_{n=1}^{\infty}\underbrace{\frac{n!\,(x-c)^n}{(2n-1)!!}}_{u_n}$ (observació: $n!!$ vol dir "n semi-factorial", no pas "doble factorial". Així: $5!! = 5\cdot 3\cdot 1$; $8!! = 8\cdot 6\cdot 4\cdot 2$)

Criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!\,(x-c)^{n+1}}{(2n+1)!!}}{\frac{n!\,(x-c)^n}{(2n-1)!!}}\right| =$

$\displaystyle= \lim_{n\to\infty}\frac{(n+1)\,|x-c|}{(2n+1)} = \frac{1}{2}|x-c| < 1 \Leftrightarrow |x-c| < 2.$

observem: $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots 1 = (2n+1)(2n-1)!!$

($2n+1$ senar, per tant $(2n+1)!!$ acaba en 1)

Radi conv. $R=2 \Rightarrow$ conv. abs. $\forall x \in (c-2, c+2)$.

$\cdot\, x = c+2$: $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, 2^n}{(2n-1)!!} = \sum_{n=1}^{\infty} \underset{\substack{\vee \\ 1}}{\frac{2n}{2n-1}} \cdot \underset{\substack{\vee \\ 1}}{\frac{2(n-1)}{2n-3}} \cdots \underset{\substack{\vee \\ 1}}{\frac{2\cdot 2}{3}} \cdot \underset{\substack{\vee \\ 1}}{\frac{1\cdot 2}{1}}$ div.

El terme general de la sèrie no té límit zero quan $n \to \infty$

$\cdot\, x = c-2$: div. per la mateixa raó.

Interval de conv. $x \in (c-2, c+2)$.

(h) $\displaystyle\sum_{n=0}^{\infty} \underbrace{(-1)^{n+1}(n+1)\,x^n}_{=u_n}$

Criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)\,x^{n+1}}{(-1)^{n+1}(n+1)\,x}\right| =$

$\displaystyle = \lim_{n\to\infty}\frac{n+2}{n+1}|x| = |x| < 1 \iff |x|<1$. Radi conv. $R=1$.

$\cdot\, x=1$: $\displaystyle\sum_{n=0}^{\infty} \underbrace{(-1)^{n+1}(n+1)}_{\text{no té límit zero}}$ div. ; $x=-1$: $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(-1)^n = \sum_{n=0}^{\infty} -(n+1)$ div.

Interval de conv. $x \in (-1, 1)$.

(i) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{x^{3n}}{n}}_{=u_n}$

criteri del quocient: $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{3n+3}}{n+1}}{\cancel{x^n}/n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x|^3 =$

$= |x|^3 < 1 \iff |x|^3 < 1 \Rightarrow |x| < 1$. Radi conv. $R=1$.

$x=1$: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ div; $x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{3n}}{n} = \sum_{n=1}^{\infty}(-1)^n/n$ conv. (alternada i de terme general decreixent a zero en valor absolut)

$(-1)^{3n} = [(-1)^3]^n = (-1)^n$

Interval de conv.: $x \in [-1, 1)$.