(14) Troben una sèrie de potències per a cadascuna de les funcions següents centrada en el punt que s'especifica:

(a) $f(x) = \dfrac{5}{2x-3}$, $c=-3$.

$$f(x) = \frac{5}{2(x+3)-9} = -\frac{5}{9}\,\frac{1}{1-\frac{2(x+3)}{9}} = -\frac{5}{9}\sum_{n=0}^{\infty}\left(\frac{2(x+3)}{9}\right)^n = -\frac{5}{9}\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n (x+3)^n.$$

(b) $f(x) = \dfrac{4x}{x^2+2x-3}$, $c=0$

$$x^2+2x-3=0 \iff x = \frac{-2\pm\sqrt{4+12}}{2} = \frac{-2\pm 4}{2} = \left\langle \begin{matrix} 1 \\ -3 \end{matrix}\right.$$

$$\frac{4x}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} \iff 4x = A(x+3)+B(x-1) \quad \begin{cases} x=1 \to A=1 \\ x=-3 \to B=3 \end{cases}$$

$$f(x) = \frac{1}{x-1} + \frac{3}{x+3} = -\frac{1}{1-x} + \frac{1}{1-(-x/3)} = -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\left(-1+\frac{(-1)^n}{3^n}\right)x^n$$

(c) $f(x) = \dfrac{x(1+x)}{(1-x)^2}$, $c=0$.

$$\begin{array}{r|l} x^2+x & x^2-2x+1 \\ \hline -x^2+2x-1 & 1 \\ \hline 3x-1 & \end{array}$$

$$x(1+x) = (1-x)^2\cdot 1 + 3x-1$$

$$\frac{x(1+x)}{(1-x)^2} = 1 + \frac{3x-1}{(1-x)^2}$$

$$\frac{3x-1}{(1-x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} \iff 3x-1 = A(1-x)+B \iff B=2,\ A=-3$$

$$f(x) = 1 - \frac{3}{1-x} + \frac{2}{(1-x)^2} = 1 - 3\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}(n+1)x^n = 1 + \sum_{n=0}^{\infty}(2n-1)x^n = \sum_{n=1}^{\infty}(2n-1)x^n$$

En quant:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \underset{\text{derivem}}{\Longrightarrow} \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \underset{\substack{\text{canvi índex} \\ m=n-1}}{\Longrightarrow} \sum_{m=0}^{\infty}(m+1)x^m = \frac{1}{(1-x)^2}$$

(d) $f(x) = \dfrac{1}{\sqrt{4+x^2}}$, $c=0$.

$$f(x) = \frac{1}{2\sqrt{1+x^2/2}} = \frac{1}{2}\left(1+x^2/2\right)^{-1/2}.$$

1/2

$$(1+z)^{-1/2} = 1 + \binom{-1/2}{1} z + \binom{-1/2}{2} z^2 + \cdots = 1 + \sum_{m=1}^{\infty} \binom{-1/2}{m} z^m$$

$$\binom{-1/2}{1} = -\frac{1}{2} \; ; \; \binom{-1/2}{2} = \frac{-1/2\,(-1/2-1)}{2!} = \frac{1/2\,(3/2)}{2!} = \frac{1\cdot 3}{2^2 \cdot 2!} = \frac{3!!}{2^2\cdot 2!}$$

$$\binom{-1/2}{3} = \frac{-1/2\,(-1/2-1)(-1/2-2)}{3!} = \frac{-1/2\,(-3/2)(-5/2)}{3!} = -\frac{1\cdot 3\cdot 5}{2^3 \cdot 3!} = -\frac{5!!}{2^3\cdot 3!}$$

Així: $\binom{-1/2}{m} = \frac{(-1)^m (2m-1)!!}{2^m \cdot m!} \implies (1+z)^{-1/2} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1)!!}{2^m \cdot m!} z^m$

$$f(x) = \frac{1}{2}(1+x^2/2)^{-1/2} \underset{\uparrow}{=} \frac{1}{2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1)!!}{2^m \cdot m!} \left(\frac{x^2}{2}\right)^m\right] =$$

Fem $z = \frac{x^2}{2}$

$$= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m (2m-1)!!}{2^{2m} \cdot m!} x^{2m}.$$

(e) $f(x) = \sqrt[4]{1+x}$, $c = 0$.

$$f(x) = (1+x)^{1/4} = 1 + \sum_{m=1}^{\infty} \binom{1/4}{m} x^m = 1 + \frac{1}{4}x + \sum_{m=2}^{\infty} (-1)^{m+1} \frac{1\cdot 3\cdots(4m-5)\, x^m}{4^m \cdot m!}$$

$$\binom{1/4}{1} = \frac{1}{4} \; ; \; \binom{1/4}{2} = \frac{1/4\,(1/4-1)}{2!} = \frac{1/4\,(-3/4)}{2!} = \frac{-1\cdot 3}{4^2\cdot 2!}$$

$$\binom{1/4}{3} = \frac{1/4\,(1/4-1)(1/4-2)}{3!} = \frac{1/4\,(-3/4)(-7/4)}{3!} = \frac{1\cdot 3\cdot 7}{4^3\cdot 3!}$$

Així: $\binom{1/4}{m} = (-1)^{m+1} \frac{1\cdot 3\cdots(4m-5)}{4^m \cdot m!}$ si $m \geq 2$