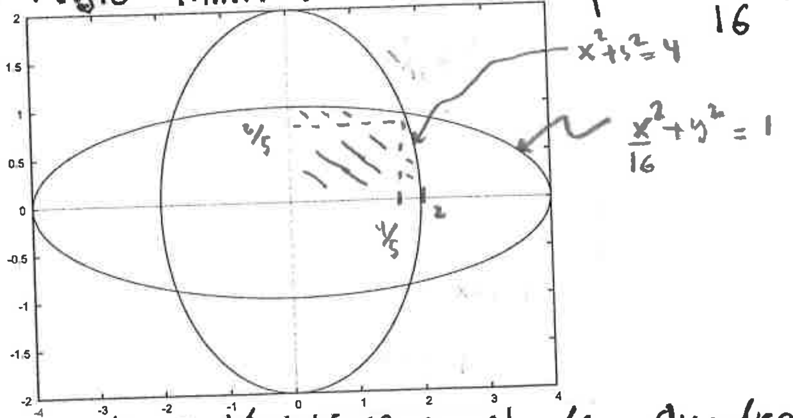
(24) Calculeu l'àrea de la intersecció del cercle limitat per $x^2+y^2=4$ i la regió limitada per l'el·lipse $\frac{x^2}{16}+y^2=1$.



• Punt de tall del cercle i l'el·lipse en el 1er. quadrant:

$$\left.\begin{array}{l} x^2+y^2=4 \\ \frac{x^2}{16}+y^2=1 \end{array}\right\} \Rightarrow x^2\left(1-\frac{1}{16}\right)=3 \Rightarrow \frac{15}{16}x^2=3 \Rightarrow x^2=\frac{16}{5} \Rightarrow x=\frac{4}{\sqrt5}$$

$$y^2=4-x^2=4-\frac{16}{5}=\frac{4}{5} \Rightarrow y=\frac{2}{\sqrt5}$$

$$\text{Àrea} = 4\,(I_1+I_2) \text{ on } I_1=\int_0^{4/\sqrt5}\sqrt{1-x^2/16}\,dx,\quad I_2=\int_{4/\sqrt5}^{2}\sqrt{4-x^2}\,dx.$$

$$I_1=\left\{\begin{array}{l} x=4\sin u \\ dx=4\cos u \end{array}\right\}=\int_0^{\arcsin(1/\sqrt5)}\sqrt{1-\sin^2 u}\;4\cos u\,du=4\int_0^{\arcsin(1/\sqrt5)}\cos^2 u\,du=$$

$$=4\int_0^{\arcsin(1/\sqrt5)}\frac{1+\cos 2u}{2}\,du=2\left[u+\frac{\sin 2u}{2}\right]_{u=0}^{u=\arcsin(1/\sqrt5)}=$$

$$=2\arcsin(1/\sqrt5)+\sin\left(2\arcsin(1/\sqrt5)\right)=2\arcsin(1/\sqrt5)+$$

$$+2\sin(\arcsin(1/\sqrt5))\cdot\cos(\arcsin(1/\sqrt5))\underset{\substack{\uparrow \\ \cos^2 u=1-\sin^2 u}}{=}2\arcsin(1/\sqrt5)+$$

$$+2\,\frac{1}{\sqrt5}\sqrt{1-\left(\frac{1}{\sqrt5}\right)^2}=2\arcsin(1/\sqrt5)+4/5$$

$$I_2=\left\{\begin{array}{l} x=2\sin u \\ dx=2\cos u\,du \end{array}\right\}=\int_{\arcsin(2/\sqrt5)}^{\pi/2}\sqrt{4-4\sin^2 u}\;2\cos u\,du=4\int_{\arcsin(2/\sqrt5)}^{\pi/2}\cos^2 u\,du=$$

$$=2\left[u+\frac{\sin 2u}{2}\right]_{u=\arcsin(2/\sqrt5)}^{u=\pi/2}=\pi-2\arcsin(2/\sqrt5)-\sin(2\arcsin(2/\sqrt5))=$$

$= \pi - 2\arcsin(2/\sqrt{5}) - 2\sin(\arcsin(2/\sqrt{5}))\cdot\cos(\arcsin(2/\sqrt{5})) \underset{\substack{\uparrow \\ \cos^2 u = 1-\sin^2 u}}{=}$

$= \pi - 2\arcsin(2/\sqrt{5}) - 2\cdot\frac{2}{\sqrt{5}}\cdot\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2} = \pi - 2\arcsin(2/\sqrt{5}) - 4/5$

Per tant, l'àrea total és:

$\text{Àrea} = 4\pi + 8\arcsin(1/\sqrt{5}) - 8\arcsin(2/\sqrt{5})$.