7 Substitucions trigonomètriques

(a) $\int \sqrt{4-x^2}\,dx = \left\{ \begin{array}{l} x = 2\sin u \\ dx = 2\cos u\,du \end{array} \right\} = \int 2\sqrt{1-\sin^2 u}\; 2\cos u\,du =$

$= 4\int \cos^2 u\,du = 4\int \frac{1+\cos 2u}{2}\,du = 2\left(u + \frac{\sin 2u}{2}\right) + c =$

$= 2\arcsin(x/2) + 2\sin u \cdot \cos u + c = 2\arcsin(x/2) + x\sqrt{1-x^2/4} + c$

$\cos^2 u = 1-\sin^2 u = 1 - x^2/4$

(b) $\int \frac{dx}{\sqrt{x^2-25}} = \frac{1}{5}\int \frac{dx}{\sqrt{(x/5)^2-1}} = \frac{1}{5}\cdot 5\,\operatorname{arg\,cosh}(x/5) + c = \operatorname{arg\,cosh}(x/5) + c$

(c) $\int \frac{dx}{x^2\sqrt{16-x^2}} = \left\{ \begin{array}{l} x = 4\sin t \\ dx = 4\cos t\,dt \end{array} \right\} = \int \frac{4\cos t\quad dt}{16\sin^2 t\sqrt{16\cos^2 t}} =$

$= \frac{1}{16}\int \frac{dt}{\sin^2 t} = \left\{ \begin{array}{l} u = \tan t \\ dt = \frac{du}{1+u^2} \\ \sin t = \frac{u}{\sqrt{1+u^2}} \end{array} \right\} = \frac{1}{16}\int \frac{\frac{du}{1+u^2}}{\left(\frac{u}{\sqrt{1+u^2}}\right)^2} = \frac{1}{16}\int u^2\,du =$

funció parell en $\sin t$ i $\cos t$.

$= \frac{1}{16}\int u^{-2}\,du = -\frac{1}{16}u^{-1} + c = -\frac{1}{16}\frac{1}{\tan t} + c = -\frac{1}{16}\frac{\cos t}{\sin t} + c =$

$= -\frac{1}{16}\frac{\sqrt{1-x^2/16}}{x/4} + c = -\frac{\sqrt{16-x^2}}{16x} + c$

$\cos^2 t = 1-\sin^2 t = 1 - x^2/16$

[En particular, hem vist $(\cot(t))' = \left(\frac{1}{\tan(t)}\right)' = -\frac{1}{\sin^2 t}$]

(d) $\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \left\{ \begin{array}{l} x = \sin t \\ dx = \cos t \end{array} \right\} = \int \frac{\cos^2 t}{\sin^4 t}\,dt = \int \frac{1}{\tan^2 t}\cdot\frac{1}{\sin^2 t}\,dt =$

$= -\int \cot^2(t)\cdot(\cot(t))'\,dt = -\frac{1}{3}\cot^3(t) + c = -\frac{1}{3}\frac{\cos^3 t}{\sin^3 t} + c =$

$= -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3} + c$

(e) $\int e^{2x}\sqrt{1+e^{2x}}\,dx = \dfrac{(1+e^{2x})^{3/2}}{3/2\cdot 2} + C = \dfrac{1}{3}(1+e^{2x})^{3/2} + C$

(f) $\int (x+1)\sqrt{x^2+2x+2}\ dx = \int (x+1)\sqrt{(x+1)^2+1} = \dfrac{((x+1)^2+1)^{3/2}}{3/2\cdot 2} + C =$

$= \dfrac{1}{3}(x^2+2x+2)^{3/2} + C$

(g) $\int \dfrac{\sqrt{x^2+1}}{x}\,dx = \left\{\begin{array}{l} x = \sinh(t) \\ dx = \cosh(t)dt \end{array}\right\} \underset{\uparrow}{=} \int \dfrac{\cosh(t)}{\sinh(t)}\cosh(t)\,dt =$

$\cosh^2(t) - \sinh^2(t) = 1 \Rightarrow \cosh(t) = \sqrt{1+x^2}$

$= \int\left(\dfrac{1}{\sinh(t)} + \sinh(t)\right) dt = \int \dfrac{dt}{\sinh(t)} + \cosh(t) \cancel{+} C$

Llavors:

$\int \dfrac{dt}{\sinh(t)} = \int \dfrac{2}{e^t - e^{-t}}\,dt = 2\int \dfrac{e^t}{e^{2t}-1}\,dt = \left\{\begin{array}{l} u = e^t \\ du = e^t\,dt \end{array}\right\} =$

$= 2\int \dfrac{u}{u^2-1}\,\dfrac{du}{u} = 2\int \dfrac{du}{u^2-1} \underset{\uparrow}{=} \int\left(\dfrac{1}{u-1} - \dfrac{1}{u+1}\right) du = \ln(u-1) - \ln(u+1) + C$

descomposició fraccions simples

$= \ln(e^t-1) - \ln(e^t+1) + C \underset{\uparrow}{=} \ln(\sinh(t) + \cosh(t) - 1) -$

$\sinh(t) = \dfrac{e^t - e^{-t}}{2}$

$\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

$-\ln(\sinh(t) + \cosh(t) + 1) = \ln(\sinh(t) + \cosh(t) - 2) - \ln(\sinh(t) + \cosh(t) + 2)$

$= \ln(x + \sqrt{1+x^2} - 1) \quad - \ln(x + \sqrt{1+x^2} + 1) \quad + C$

Finalment: $\int \dfrac{\sqrt{x^2+1}}{x}\,dx = \ln(x+\sqrt{1+x^2}-1) - \ln(x+\sqrt{1+x^2}+1) + \sqrt{1+x^2}$