⑥ Funcions racionals.

(a) $\displaystyle\int \frac{5x}{x^2-10x+25}\,dx$

$x^2-10x+25=(x-5)^2$

$\dfrac{5x}{(x-5)^2}=\dfrac{A}{x-5}+\dfrac{B}{(x-5)^2} \Rightarrow 5x=A(x-5)+B \Rightarrow A=5,\ B=25$

Així: $\displaystyle\int \frac{5x}{x^2-10x+25}\,dx = 5\int\frac{dx}{x-5}+25\int(x-5)^{-2}dx = 5\ln|x-5|+25\frac{(x-5)^{-1}}{-1}+C$

$= 5\ln|x-5|-\dfrac{25}{x-5}+C.$

(b) $\displaystyle\int \frac{1}{x^2-25}\,dx$

$x^2-25=(x-5)(x+5)$

$\dfrac{1}{(x-5)(x+5)}=\dfrac{A}{x-5}+\dfrac{B}{x+5} \Rightarrow 1=A(x+5)+B(x-5)$ $\begin{cases} x=5 \to A=\frac{1}{10} \\ x=-5 \to B=-\frac{1}{10}\end{cases}$

$\displaystyle\int\frac{dx}{x^2-25}=\int\frac{1/10}{x-5}\,dx+\int\frac{-1/10}{x+5}\,dx=\frac{1}{10}\ln|x-5|-\frac{1}{10}\ln|x+5|+C$

(c) $\displaystyle\int \frac{x+1}{(x^2+4x+5)^2}\,dx$

$x^2+4x+5=0 \Leftrightarrow x=\dfrac{-4\pm\sqrt{16-25}}{2}$ no té arrels reals!

$\dfrac{x+1}{(x^2+4x+5)^2}=\dfrac{Ax+B}{x^2+4x+5}+\dfrac{Cx+D}{(x^2+4x+5)^2}$ d'on: $A=B=0$, $C=1, D=1$

A més, $x^2+4x+5=(x+2)^2+1$

Llavors:

$\displaystyle\int\frac{x+1}{(x^2+4x+5)^2}\,dx=\int\frac{x+2}{((x+2)^2+1)^2}\,dx+\int\frac{-1}{((x+2)^2+1)^2}\,dx$

La 1a integral és immediata. Per calcular la segona, el punt clau és saber calcular:

$\displaystyle\int\frac{dx}{(1+x^2)^2}=\left\{\begin{array}{l} x=\tan u\\ dx=(1+\tan^2 u)\,du\end{array}\right\}=\int\frac{du}{1+\tan^2 u}=\int\cos^2 u\,du=$

$\displaystyle=\int\left(\frac{1}{2}+\frac{\cos 2u}{2}\right)du=\frac{u}{2}+\frac{\sin 2u}{4}+C=\frac{u}{2}+\frac{\sin u\cos u}{2}+C$

Llavors usant $\left.\begin{array}{l}\cos u \cdot x - \sin u = 0 \\ \cos^2 u + \sin^2 u = 0\end{array}\right\}$ $\cos u = \frac{1}{\sqrt{1+x^2}}$, $\sin u = \frac{x}{\sqrt{1+x^2}}$

Per tant : $\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + C$

Així:

$$\int \frac{x+1}{(x^2+4x+5)^2}\,dx = \frac{[(x+2)^2+1]^{-1}}{(-1)\cdot 2} - \frac{1}{2}\arctan(x+2) - \frac{x+1}{2[(x+1)^2+2]} + C$$

(d) $\int \frac{x}{(a+bx)^2}\,dx$ $(b \neq 0)$

$$\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2} \Rightarrow x = A(a+bx) + B \Rightarrow \left\{\begin{array}{l} A = 1/b \\ B = -a/b\end{array}\right.$$

Així: $\int \frac{x}{(a+bx)^2}\,dx = \int \frac{1/b}{a+bx}\,dx + \int \frac{-a/b}{(a+bx)^2}\,dx = \frac{1}{b^2}\ln|a+bx| -$

$$- \frac{a}{b}\frac{(a+bx)^{-1}}{(-1)\,b} + C = \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\frac{1}{a+bx} + C.$$

(e) $\int \frac{1}{x^2(a+bx)}\,dx$ $(a \cdot b \neq 0)$

$$\frac{1}{x^2(a+bx)} = \frac{A}{a+bx} + \frac{B}{x} + \frac{C_1}{x^2} \Rightarrow 1 = A\cdot x^2 + Bx(a+bx) + C_1(a+bx)$$

d'on: $aC_1 = 1$, $B\cdot a + C_1 b = 0$, $A + bB = 0 \Rightarrow \left\{\begin{array}{l} C_1 = 1/a,\ B = -b/a^2 \\ A = b^2/a^2\end{array}\right.$

Així: $\int \frac{dx}{x^2(a+bx)} = \frac{b}{a^2}\ln|a+bx| - \frac{b}{a^2}\ln|x| - \frac{1}{a}\frac{1}{x} + C_1$

(f) $\int \frac{x^2+x+1}{x(x^2+1)}\,dx$

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow x^2+x+1 = A(x^2+1) + (Bx+C)x$$

d'on: $A=1$, $C=1$, $A+B=1$ $\Rightarrow$ $A = C = 1$, $B = 0$.

Així: $\int \frac{x^2+x+1}{x(x^2+1)}\,dx = \ln|x| + \arctan(x)$

(g) $\int \frac{1}{x^3(x^3+1)}\,dx$

$x^3(x^3+1) = x^3(x+1)(\underbrace{x^2-x+1})$ ← sense arrels reals.

Ruffini:

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

Així: $\frac{1}{x^3(x^3+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2-x+1} \Rightarrow$

$\Rightarrow 1 = Ax^2(x^3+1) + Bx(x^3+1) + C(x^3+1) + Dx^3(x^2-x+1) + (Ex+F)x^3(x+1).$

$C=1,\ B=0,\ A=0,\ C+D+F=0,\ B-D+E+F=0,\ A+D+E=0$

d'on: $A=0,\ B=0,\ C=1,\ D=-\frac{1}{3},\ E=\frac{1}{3},\ F=-2/3$, ja que:

$$\left.\begin{array}{l} D+F=-1 \\ -D+E+F=0 \\ D+E=0 \end{array}\right\} \begin{array}{l} (eq_1)+(eq_2): E+2F=-1 \\ (eq_2)+(eq_3): 2E+F=0 \end{array} \left.\right\} \begin{array}{l} F=-2E \\ -3E=-1 \to E=1/3 \to F=-2/3 \Rightarrow D=-\frac{1}{3} \end{array}$$

Així, tot es redueix al càlcul de:

$\int \frac{\frac{1}{3}x - 2/3}{x^2-x+1}\,dx = \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \int \frac{1/6 - 2/3}{x^2-x+1}\,dx = \frac{1}{6}\ln|x^2-x+1| -$

$-\frac{1}{2}\int \frac{dx}{(x-1/2)^2+3/4} = \frac{1}{6}\ln|x^2-x+1| - \frac{2}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} = \frac{1}{6}\ln|x^2-x+1| -$

$-\frac{2}{3}\frac{\sqrt{3}}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$

Finalment:

$\int \frac{dx}{x^3(x^3+1)} = -\frac{x^{-2}}{2} - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$