## 2 Canvi de Variables

(a) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \int x(x^2+1)^{-1/2}dx = (x^2+1)^{1/2} + c = \sqrt{x^2+1} + c.$

(b) $\displaystyle I = \int \frac{1+x}{1+\sqrt{x}}\,dx = \left\{\begin{array}{l} x = u^2 \\ dx = 2u\,du \end{array}\right\} = \int \frac{1+u^2}{1+u}2u\,du = 2\int \frac{u+u^3}{1+u}\,du$

$$\begin{array}{l|l} u^3+u & u+1 \\ \hline -u^3-u^2 & u^2-u+2 \\ \hline -u^2+u & \\ u^2+u & \\ \hline 2u & \\ -2u-2 & \\ \hline -2 & \end{array}$$

i $u^3+u = (u+1)(u^2-u+2) - 2$

$$\frac{u+u^3}{1+u} = u^2-u+2 - \frac{2}{1+u}$$

Així: $\displaystyle I = 2\int \left(u^2-u+2-\frac{2}{1+u}\right)du =$

$\displaystyle = 2\left[\frac{u^3}{3} - \frac{u^2}{2} + 2u - 2\ln(1+u)\right] + c =$

$\displaystyle = 2\frac{x^{3/2}}{3} - x + 4x^{1/2} - 4\ln(1+\sqrt{x}) + c.$

(c) $\displaystyle\int \frac{\ln 2x}{x\ln 4x}\,dx = \left\{\begin{array}{l} u = \ln 4x \\ du = dx/x \\ x = e^u/4 \end{array}\right\} = \int \frac{\ln(e^u/2)}{x\cdot u}\,x\,du = \int \frac{u-\ln 2}{u}\,du =$

$\displaystyle = \int\left(1 - \frac{\ln 2}{u}\right)du = u - \ln 2\cdot\ln u + c = \ln 4x - \ln 2\cdot\ln(\ln 4x) + c$

(d) $\displaystyle\int x\sqrt[3]{3-4x^2}\,dx = \int x(3-4x^2)^{1/3}dx = \frac{(3-4x^2)^{4/3}}{4/3\cdot(-8)} + c = -\frac{3}{32}(3-4x^2)^{4/3} + c$

(e) $\displaystyle\int \sec(2x)\tan(2x)\,dx = \int \sin(2x)\cdot(\cos(2x))^{-2}dx = \frac{(\cos(2x))^{-1}}{(-1)(-2)} + c =$

$\displaystyle = \frac{1}{2\cos(2x)} + c$

(f) $\displaystyle\int \frac{x-\sqrt{\arctan x}}{1+x^2}\,dx = \left\{\begin{array}{l} u = \arctan(x) \\ du = \dfrac{dx}{1+x^2} \end{array}\right\} = \int(\tan u - \sqrt{u})\,du =$

$\displaystyle = \int\left(\frac{\sin u}{\cos u} - u^{1/2}\right)du = -\ln|\cos u| - \frac{u^{3/2}}{3/2} + c = -\ln|\cos(\operatorname{atan} x)| -$

$\displaystyle -\frac{2}{3}(\operatorname{atan}(x))^{3/2} + c.$

A més: $\displaystyle x^2 = \tan^2(\operatorname{atan} x) = \frac{\sin^2(\operatorname{atan} x)}{\cos^2(\operatorname{atan} x)}$ i $\sin^2(\operatorname{atan} x) + \cos^2(\operatorname{atan} x) = 1$

Així: $\cos(\mathrm{atan}\,x) = \dfrac{1}{\sqrt{1+x^2}}$ i $\sin(\mathrm{atan}\,x) = \dfrac{x}{\sqrt{1+x^2}}$.

Per tant
$$\int \frac{x-\sqrt{\mathrm{atan}\,x}}{1+x^2}\,dx = -\ln\left(\frac{1}{\sqrt{1+x^2}}\right) - \frac{2}{3}(\mathrm{atan}(x))^{3/2} + C$$
$$= \frac{1}{2}\ln(1+x^2) - \frac{2}{3}(\mathrm{atan}(x))^{3/2} + C.$$

(g)
$$\int \frac{ax+b}{a^2x^2+b^2}\,dx = \frac{1}{2a}\int \frac{2a^2x}{a^2x^2+b^2}\,dx + \frac{1}{a}\int \frac{a/b}{\left(\frac{ax}{b}\right)^2+1}\,dx =$$
$$= \ln(a^2x^2+b^2) + \mathrm{atan}\left(\frac{ax}{b}\right) + C.$$

(h)
$$\int \frac{e^x}{e^x-1}\,dx = \ln(e^x-1) + C$$

(i)
$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\sin(\sqrt{x}) + C$$

(j)
$$\int \frac{a^{2x}-1}{\sqrt{a^x}}\,dx = \left\{\begin{array}{l} u = a^{\frac{1}{2}x} = e^{\frac{1}{2}\ln a\cdot x} \\ du = \frac{1}{2}\ln a\; a^{\frac{1}{2}x}\,dx \end{array}\right\} = \int \frac{u^4-1}{u}\,\frac{du}{\frac{1}{2}\ln a\; u} =$$
$$= \frac{2}{\ln a}\int (u^2-u^{-2})\,du = \frac{2}{\ln a}\left(\frac{u^3}{3} - \frac{u^{-1}}{-1}\right) + C = \frac{2}{\ln a}\left(\frac{a^{\frac{3}{2}x}}{3} + \frac{1}{\sqrt{a^x}}\right) + C$$

( també: $\displaystyle\int (a^{3/2x} - a^{-x/2})\,dx = \int (e^{(3/2\ln a)x} - e^{-\frac{\ln a}{2}x})\,dx = \frac{2}{3\ln a}a^{3/2x} + \frac{2}{\ln a}a^{-x/2} + C$ )

(k)
$$\int \frac{\sqrt[3]{1+\ln x}}{x}\,dx = \int \frac{1}{x}(1+\ln x)^{1/3}\,dx = \frac{(1+\ln x)^{4/3}}{4/3} + C$$

(l)
$$\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx = \int \left[(2-x^2)^{-1/2} - (2+x^2)^{-1/2}\right]dx = \left\{\begin{array}{l} x = \sqrt{2}u \\ dx = \sqrt{2}\,du \end{array}\right\} =$$
$$= \frac{1}{\sqrt{2}}\int \left(\frac{1}{\sqrt{1-u^2}} - \frac{1}{\sqrt{1+u^2}}\right)du = \frac{1}{\sqrt{2}}\left(\arcsin(u) - \mathrm{argsinh}(u)\right) + C =$$
$$= \frac{1}{\sqrt{2}}\left(\arcsin(x/\sqrt{2}) - \mathrm{argsinh}(x/\sqrt{2})\right) + C.$$