41 Escriviu la fórmula de MacLaurin fins l'ordre 3 de la funció.

$$f(x) = \sqrt{1+\sqrt{1+x}}$$

- opció 1: calcular derivades de $f(x) = \left(1+(1+x)^{1/2}\right)^{1/2}$

$$f'(x) = \frac{1}{4}\left(1+(1+x)^{1/2}\right)^{-1/2}\cdot(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{16}\left(1+(1+x)^{1/2}\right)^{-3/2}\cdot(1+x)^{-1} - \frac{1}{8}\left(1+(1+x)^{1/2}\right)^{-1/2}\cdot(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{64}\left(1+(1+x)^{1/2}\right)^{-5/2}(1+x)^{-3/2} + \frac{3}{32}\left(1+(1+x)^{1/2}\right)^{-3/2}\cdot(1+x)^{-2} + \frac{3}{16}\left(1+(1+x)^{1/2}\right)^{-1/2}(1+x)^{-5/2}$$

$f(0) = \sqrt{2}$, $f'(0) = \frac{1}{4}\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{8}$; $f''(0) = -\frac{1}{16}\frac{1}{(\sqrt{2})^3} - \frac{1}{8}\cdot\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{64}$

$$f'''(0) = \frac{3}{64}\cdot\frac{1}{(\sqrt{2})^5} + \frac{3}{32}\cdot\frac{1}{(\sqrt{2})^3} + \frac{3}{16}\cdot\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{2^9}(1+2^2+2^4) = \frac{3\cdot 21\cdot\sqrt{2}}{2^9}$$

Per tant:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + R_3(x) = \sqrt{2} + \frac{1}{8}x - \frac{5\sqrt{2}}{128}x^2 + \frac{21\sqrt{2}}{1024}x^3 + R_3(x)$$

- opció 2: Per generació

Recordem: $(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + R_3(x)$

Si $\alpha = 1/2$: $\binom{1/2}{1} = 1/2$, $\binom{1/2}{2} = \frac{1/2(1/2-1)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{1/2(1/2-1)(1/2-2)}{3!} = \frac{1}{16}$

Així: $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o_4(x)$

$$f(x) = \left(2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o_4(x)\right)^{1/2} = \sqrt{2}\left(1 + \frac{x}{4} - \frac{x^2}{16} + \frac{x^3}{32} + o_4(x)\right)^{1/2} =$$

$$= \sqrt{2}\left[1 + \frac{1}{2}\left(\frac{x}{4} - \frac{x^2}{16} + \frac{x^3}{32} + o_4(x)\right) - \frac{1}{8}\left(\frac{x}{4} - \frac{x^2}{16} + o_3(x)\right)^2 + \frac{1}{16}\left(\frac{x}{4} + o_2(x)\right)^3 + o_4(x)\right] =$$

$$= \sqrt{2}\left[1 + \frac{x}{8} - \frac{x^2}{32} + \frac{x^3}{64} - \frac{1}{8}\left(\frac{x^2}{16} - \frac{x^3}{32}\right) + \frac{1}{16}\frac{x^3}{64} + o_4(x)\right] =$$

$$= \sqrt{2}\left[1 + \frac{x}{8} - \frac{5}{128}x^2 + \frac{21}{1024}x^3 + o_4(x)\right]$$

D'on tornem a obtenir el mateix desenvolupament per f(x) !