

Notes on symplectic homology

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In this document we review the theory of symplectic homology, particularly its construction from Floer homology.

This is a summary of Section 6 of *Floer homologies, with applications* by Alberto Abbondandolo and Felix Schlenk.

1 Liouville domains

Definition 1 Let (M, ω) a compact symplectic manifold with boundary, and let ω be exact, so that $\omega = d\lambda$. Let $X_\lambda \in \mathfrak{X}(M)$ be a Liouville vector field, this means, such that

$$i_{X_\lambda} \omega = \lambda \Leftrightarrow \mathcal{L}_{X_\lambda} \omega = \omega.$$

We say that (M, λ) is a **Liouville domain** if X_λ is transverse to the boundary ∂M .

If $j_{\partial M}$ denotes the natural inclusion of ∂M into M , then we will denote $\alpha = j_{\partial M}^* \lambda \in \Omega^1(\partial M)$.

Remark 2 Under these assumptions $(\partial M, \alpha)$ is a contact manifold.

Remark 3 Let $\mathcal{LM}$ denote the space of loops $\{x : S^1 \rightarrow M \mid x \text{ is smooth}\}$ for any exact¹ symplectic manifold $(M, \omega = d\lambda)$. Then, the action functional has the expression

$$\begin{aligned} \mathcal{A}_H : \mathcal{LM} &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \int_{S^1} \left(-H(\gamma(t)) + \lambda_{\gamma(t)} \left(\frac{d\gamma}{dt}(t) \right) \right) dt . \end{aligned}$$

Definition 4 (Completion): As X_λ is a Liouville vector field, we know that $(\varphi_{X_\lambda}^s)^* \omega = e^s \omega$. Therefore, as $\varphi_{X_\lambda}^s$ is defined for all $s \leq 0$, we have a natural embedding

$$\begin{aligned} j : (0, 1] \times \partial M &\longrightarrow M \\ (\rho, x) &\longmapsto \varphi_{X_\lambda}^{\log \rho}(x) . \end{aligned}$$

Then, $j^* \lambda = \rho \alpha$ and $j^* X_\lambda|_{\text{Im} j} = \rho \partial \rho$.

The **completion** of (M, λ) is the exact symplectic manifold $\hat{M} = M \cup_{\partial M} ([1, +\infty) \times \partial M)$, where the Liouville form can be naturally extended by

$$\lambda|_{[1, +\infty) \times \partial M} = \rho \alpha.$$

Intuitively, $(\hat{M}, \lambda)$ is obtained by gluing along ∂M the symplectization of the contact manifold $(\partial M, \alpha)$.

¹As the symplectic manifold is exact, we do not require the space to be composed of contractible loops as usual.

2 Construction of the symplectic homology

Let $H : \hat{M} \rightarrow \mathbb{R}$ a Hamiltonian vanishing on M and such that its restriction to $\hat{M} \setminus M$ depends only on the coordinate ρ ,

$$H = h(\rho) \quad \forall \rho \geq 1.$$

Here, h is a function from $[1, +\infty)$ to $\mathbb{R}$ such that

1. $h(1) = h'(1) = 0$.
2. $h''(\rho) > 0 \quad \forall \rho > 1$.
3. $\lim_{\rho \rightarrow \infty} h'(\rho) = +\infty$.

Under these conditions, we are interested on the dynamics of the Hamiltonian vector field X_H . First of all, it is obvious that $X_H|_M = 0$, so all the points of M are fixed points.

Claim 1 On $\hat{M} \setminus M = (1, +\infty) \times \partial M$, the Hamiltonian vector field has the form

$$X_H(\rho, x) = h'(\rho)R_\alpha(x),$$

where $R_\alpha \in \mathfrak{X}(\partial M)$ is the Reeb vector field of $(\partial M, \alpha)$.

Proof. First, notice that $dH = h'(\rho)d\rho$, and let $X_H = a\partial_\rho + Y$, where a is a function and $Y \in \mathfrak{X}(\partial M)$.
As

$$\omega = d\lambda = d\rho \wedge \alpha + \rho d\alpha,$$

We get that

$$i_{X_H}\omega = a\alpha - \alpha(Y)d\rho + \rho i_Y d\alpha.$$

Thus, as $i_{X_H}\omega = -dH$, we get that $a = 0$ and, moreover,

$$\begin{cases} \alpha(Y) = h'(\rho) \\ i_Y d\alpha = 0 \end{cases} \Rightarrow X_H = Y = h'(\rho)R_\alpha.$$

□

Therefore, for each periodic orbit γ of R_α with period $T > 0$, the equation

$$h'(\rho) = T$$

has a unique solution $\rho_0 > 1$. Then, the orbit

$$\bar{\gamma}(t) = (\rho_0, \gamma(Tt))$$

is a 1-periodic orbit of X_H . The correspondence, moreover, is bidirectional, so we know that for each periodic orbit of R_α there is a unique 1-periodic orbit of X_H .

Claim 2 For such periodic orbits,

$$\mathcal{A}_H(\bar{\gamma}) = \rho_0 h'(\rho_0) - h(\rho_0).$$

Proof. If we derive, we get that

$$\dot{\bar{\gamma}}(t) = (0, T\gamma(Tt)) = TR_\alpha(\gamma(Tt)) = h'(\rho_0)R_\alpha(\gamma(Tt)).$$

Recall also that, in the completion, $\lambda = \rho\alpha$. Using the expression of the action functional in Remark 3, we get that

$$\mathcal{A}_H(\bar{\gamma}) = \int_0^1 (-h(\rho_0) + \lambda_{\bar{\gamma}}(\dot{\bar{\gamma}})) dt = -h(\rho_0) + \int_0^1 h'(\rho_0) i_{R_\alpha}(\rho_0 \alpha) dt = -h(\rho_0) + \rho_0 h'(\rho_0).$$

□

It is interesting to note that

$$\frac{d}{d\rho} (\rho h'(\rho) - h(\rho)) = \rho h''(\rho) > 0,$$

so the correspondence is a positive and strictly increasing function.

Therefore, the correspondence Period $\mapsto$ Action is strictly monotone (and thus one to one).

There are two obstacles we must overcome before being able to define a Floer complex over these periodic orbits:

1. H is a highly degenerate function in the sense that all the points of M are degenerate (and furthermore non isolated) periodic orbits. We must guarantee that these degenerate periodic orbits do not appear in our construction.
2. $\hat{M}$ is a non-compact space, so it is impossible to guarantee that there is a finite number of generators².

Both problems can be avoided using a small perturbation of H by a C^2 -small periodic function we will not describe the construction here, but we will point at the consequences of such construction.

Problem 1: If we assume that R_α is non-degenerate in the sense that the linearization of its flow at the periodic orbits does not have 1 as an eigenvalue, then the perturbation of H achieves the intended purpose. However, after the perturbation the correspondence between periodic orbits of R_α and X_H is no longer one-to-one. Instead, for every closed orbit of R_α we will have exactly two 1-periodic orbits of X_H close to $\bar{\gamma}$ the 1-periodic orbit corresponding to the flow of the unperturbed Hamiltonian.

Problem 2: Even if $\hat{M}$ is non-compact, the set of critical points of $\mathcal{A}_H$ intersected with any sublevel set $\{\mathcal{A}_H \leq a \mid a > 0\}$ is finite. A way to interpret this is that imposing an upper bound on the period of the Reeb orbits guarantees finiteness on the number of orbits.

The only thing that remains to prove is that $\mathcal{M}_{H,J}(\gamma, ta)$ is compact for all periodic orbits x and y . To do this, it is enough to prove that the images of the elements of $\mathcal{M}_{H,J}(\gamma, ta)$ are contained in a compact subset of $\hat{M} \setminus M$.

To this end, it will be enough to prove that for any solution of the Floer equation u , $\tilde{\rho} := \rho \circ u$ is bounded. This, however, is true because

$$\Delta \tilde{\rho} = \|\partial_s u\|_{J_t}^2 + \tilde{\rho} h''(\tilde{\rho}) \partial_s \tilde{\rho} > 0,$$

(in the region where $\rho \geq \rho_0$), and we can apply the maximum principle.

Therefore, we can define a Floer complex with the periodic orbits of the Hamiltonian flow of the perturbed $\tilde{H}$. We will denote this complex by $(\text{CF}_k(H), \partial_k)$. It is important to highlight that, even though $\text{CF}_h(H)$ may be infinitely generated, $\partial_k \gamma$ is always well defined because the set $\mathcal{M}_{H,J}(\gamma, \eta)$ is non-empty only for a finite number of η 's (because $\mathcal{A}_H(\eta) \leq \mathcal{A}_H(\gamma)$ and critical points within sublevel sets of $\mathcal{A}_H$ are finite).

Therefore, the homology of the complex may be defined. With this, we are well equipped to formalize the object of study of this document

Definition 5 The **symplectic homology** of the Liouville domain (M, λ) is

$$\text{SH}_\bullet(M, \lambda) := \text{HF}_\bullet(\hat{M}).$$

²This is not necessarily an obstacle to generate the groups of the complex, but to define the boundary map.

As implied by the notation, this homology depends not only on the topology of M , but also on the Liouville form λ .

However, the homology is invariant under smooth perturbations of the symplectic form under the condition that ∂M remains of contact type throughout the whole deformation.

3 Symplectic homology homomorphism

Proposition 6 *There is a natural homomorphism from the singular relative homology to the symplectic homology of a Liouville domain,*

$$c_\bullet : H_{\bullet+n}(M, \partial M) \longrightarrow SH_\bullet(M, \lambda) .$$

Proof. Let us start by recalling that, for the function H constructed at the beginning of Section 2, the critical points of the action functional belong to either of these two sets:

1. Fixed points with action 0. These are all the points of M .
2. Non-constant loops in $\hat{M} \setminus M$ with strictly positive action. Let $\varepsilon > 0$ such that $\mathcal{A}_H(\gamma) > 2\varepsilon$ for all loops belonging to this category.

As $SH_\bullet(M, \lambda)$ does not depend on the Hamiltonian H , we can take a different perturbation, which we will denote by $\tilde{H}$, such that

1. $\tilde{H}|_{\partial M} = 0$.
2. $\tilde{H}|_{\hat{M}} < 0$.
3. $\tilde{H}$ is C^2 -small on M .

Then, the critical points of $\mathcal{A}_{\tilde{H}}$ inside of M are isolated fixed points and they can be taken with action strictly lower than ε .

Let us denote by $K_\bullet$ the subcomplex of $CF_\bullet(\hat{M})$ generated by the critical points with action strictly less than ε . As the boundary map of the Floer complex follows the path of the negative gradient, $(K_\bullet, \partial_\bullet)$ is a well defined subcomplex.

Moreover, $K_\bullet$ is precisely the Morse complex of $-\tilde{H}$ (with an index shift of n), so the homology of this complex is the same as the Morse homology of the manifold with boundary $(M, \partial M)$ with the gradient transverse to the boundary, and this homology is isomorphic to the relative singular homology.

Then, the inclusion $K_\bullet \hookrightarrow CF_\bullet(\hat{M}, \tilde{H})$ induces the map $c_\bullet : H_{\bullet+n}(M, \partial M) \rightarrow SF_\bullet(M, \lambda)$ at the homology level. $\square$

Remark 7 It is relevant to point out that the generators left out of $K_\bullet$ are precisely the ones corresponding to the periodic orbits of the Reeb vector field on the boundary. Therefore, we can already establish the following result:

Theorem 8 *Let (M, λ) a Liouville domain. If the map $c_\bullet : H_{\bullet+n}(M, \partial M) \rightarrow SH_\bullet(M, \lambda)$ is not an isomorphism, then the Weinstein conjecture is true for the contact manifold $(\partial M, \alpha = j_M^* \lambda)$, this means, the Reeb vector field R_α has at least one periodic orbit.*

Example: For any starshaped domain $M \subset \mathbb{R}^{2n}$, the symplectic homology $SH_\bullet$ vanishes, whereas $H_\bullet(M, \partial M) \neq 0$. Thus, Weinstein's conjecture is true for (M, λ_{st}) .

4 Viterbo functoriality

Definition 9 A **Liouville embedding** is an embedding of Liouville domains $i : (M', \lambda') \rightarrow (M, \lambda)$ of codimension 0 such that

$$i^* \lambda = c \lambda' + df$$

for some $c > 0$ and $f \in C^\infty(M', \mathbb{R})$.

A Liouville embedding induces a homomorphism $SH_\bullet(i) : SH_\bullet(M, \lambda) \rightarrow SH_\bullet(M', \lambda')$. Moreover, we have functoriality with respect to the homomorphism $c_\bullet$,

$$\begin{array}{ccc} H_{\bullet+n}(M, \partial M) & \xrightarrow{i!} & H_{\bullet+n}(M', \partial M') \\ c_\bullet \downarrow & & \downarrow c_\bullet \\ SH_\bullet(M, \lambda) & \xrightarrow{SH_\bullet(i)} & SH_\bullet(M', \lambda') \end{array}$$

where $i! : H_{\bullet+n}(M, \partial M) \cong H^{n-\bullet}(M) \xrightarrow{i^*} H^{n-\bullet}(M') \cong H_{\bullet+n}(M', \partial M')$.

5 Obstructions to symplectic embeddings

Definition 10 A **symplectic capacity** is a correspondence

$$C : \{\text{symplectic manifolds of dimension } 2n\} \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

such that it has the following properties:

1. **Monotonicity:** If there exists some $\varphi : (M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ symplectic embedding, then

$$C(M_1, \omega_1) \leq C(M_2, \omega_2).$$

2. **Scaling:** $C(M, k\omega) = |k|C(M, \omega)$.

3. **Normalization:** If we denote

$$B(r) = \{x \in \mathbb{R}^{2n} \mid |x| = r\}$$

$$Z(r) = \{x \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 = r^2\}$$

then $C(B(1), \omega_{\text{st}}) = C(Z(1), \omega_{\text{st}}) = \pi$.

We will see that the symplectic homology can be used to define something analogous to a symplectic capacity (using Liouville embeddings instead of symplectic embeddings).

Definition 11 A **filtering** of a Floer complex is the sequence of chain complexes $\text{CF}_\bullet^a(M, H)$ generated by the critical points

$$\text{Crit}^a(\mathcal{A}_H) = \{\gamma \in \text{Crit}(\mathcal{A}_H) \mid \mathcal{A}_H(\gamma) \leq a\}.$$

As the boundary map of the chain complex follows the flow of the negative gradient of the action functional, we conclude that $\partial_\bullet$ maps $\text{CF}_\bullet^a(M, H)$ to itself for any $a > 0$. This induces the sequence of homologies $\text{SH}_\bullet^a(M, \lambda)$ for $a > 0$.

Remark 12 The Viterbo functoriality can be refined to the filtered case.

Let us define

$$\Gamma(M, \lambda) := \inf \{a > 0 \mid c_n : H_{2n}(M, \partial M) \rightarrow \text{SH}_n^a(M, \lambda) \text{ vanishes}\} \in [0, +\infty].$$

Remark 13 If the morphism $c_\bullet$ vanishes, then $\Gamma(M, \lambda) < +\infty$.

Remark 14 The correspondence Γ is monotone with respect to Liouville embeddings.

Remark 15 The maps $c_\bullet$ are isomorphisms on the filtration as long as a is smaller than the action of the shortest Reeb orbit on ∂M , which we can denote by $\gamma_{\min}$. Thus, $\Gamma(M, \lambda) \geq \mathcal{A}_H(\gamma_{\min})$.

As an interesting application, this “Liouville capacity” can be used to prove Gromov’s non-squeezing theorem:

Theorem 16 (Gromov): *Let us assume that there exists a symplectic embedding $\varphi : B(a) \rightarrow Z(b)$. Then, $a \leq b$.*

Lemma 17 *Let $E(a_1, \dots, a_n)$ denote the Liouville domain*

$$E(a_1, \dots, a_n) = \{x \in \mathbb{R}^{2n} \mid x_i^2 + y_i^2 \leq a_i^2 \ 1 \leq i \leq n\},$$

with the convention that $a_1 \leq a_2 \leq \dots \leq a_n$. Then,

$$\Gamma(M, \lambda_{\text{st}}) = a_1.$$

Proof. (Theorem): Our hypothesis is that we have an embedding $\varphi : B(a) \rightarrow Z(b)$. Take $\varepsilon > 0$, and let us construct the ellipsoids

- $E = E(a_1, \dots, a_n)$, with

$$a - \varepsilon < a_1 < \dots < a_n < a.$$

- $E' = E(b_1, \dots, b_n)$, with

$$b = b_1 < \dots < b_n.$$

And in such a way that $\varphi(E) \subset E'$, so it is a Liouville embedding.

Then, by our lemma, $\Gamma(E) = a_1$ and $\Gamma(E') = b$, and by monotonicity $\Gamma(E) \leq \Gamma(E')$.

Thus,

$$a - \varepsilon < b \ \forall \varepsilon > 0,$$

so, as a consequence,

$$a \leq b.$$

□