

Talk notes on b^k -symplectic geometry

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These are the notes of the talk done for the seminar of Geometry and Analysis at the University of Antwerp on 25/06/2019. Here we intend to provide an introduction to b^k -geometry, and the construction of symplectic structures in this context.

1 Motivation

This is an example taken from [3]. Consider the restricted planar three body problem on $\mathbb{R}^2$, which in the context of symplectic geometry is given by the standard symplectic form on $T^*\mathbb{R}^2 \cong \mathbb{R}^4$,

$$\omega_{st} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2,$$

and the Hamiltonian given by the mechanical energy of the system,

$$H(q, p) = E_{kinetic} + E_{potential} = \frac{\|p\|^2}{2} + \frac{1 - \mu}{\|q - q_1\|} + \frac{\mu}{\|q - q_2\|}.$$

If we transform this system to polar coordinates $(q_1, q_2) = (r \cos \alpha, r \sin \alpha)$ (and the canonical change to the momenta), the Hamiltonian gets transformed to

$$H(r, \alpha, P_r, P_\alpha) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} + U(r \cos \alpha, r \sin \alpha),$$

In this context, we might apply the McGehee change of coordinates to study the behaviour of orbits “escaping” to infinity:

$$r = \frac{2}{x^2}.$$

If do not impose the condition that the change of coordinates is canonical (this means, we allow it to change the symplectic structure), we see that the symplectic form gets transformed to

$$\omega = \frac{4}{x^3} dx \wedge dP_r + d\alpha \wedge dP_\alpha,$$

which is not defined when $x = 0$. However, this structure can be made sense of in the context of b^3 -symplectic geometry.

2 b -geometry

2.1 The b -tangent bundle

A thorough introduction to b -geometry can be found in the second chapter of [2], although here we follow the scheme of [1].

In what follows, M will denote a smooth manifold of dimension n , and Z will denote a smooth hypersurface $Z \subset M$. Let ${}^b\mathfrak{X}(M)$ denote the set of vector fields which are tangent to Z . It can be easily seen that this is a Lie subalgebra of $\mathfrak{X}(M)$.

Remark 2.1 Consider $(U, (x_1, \dots, x_n))$ a local chart such that $Z \cap U = \{x_1 = 0\}$. Then, we can see that ${}^b\mathfrak{X}(U) = \langle x_1 \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle$.

Let us recall the following theorem from [5]:

Theorem 2.2 Consider P a $\mathcal{C}^\infty(M)$ module. There exists a finite dimensional vector bundle $E \xrightarrow{\pi} M$ such that $P = \Gamma(E)$ if, and only if, P is projective and finitely generated.

In this context, it is clear that ${}^b\mathfrak{X}(M)$ satisfies the conditions of the theorem, so there exists a vector bundle that has the elements of this subalgebra as sections:

Definition 2.3 The *b-tangent bundle* bTM is the vector bundle such that $\Gamma({}^bTM) = {}^b\mathfrak{X}(M)$.

Remark 2.4 It can be seen that, fiberwise,

$$\begin{aligned} {}^bT_p M &\cong \begin{cases} \langle x_1 \partial_{x_1} |_p \rangle \oplus T_p Z & \text{if } p \in Z \\ T_p M & \text{if } p \notin Z \end{cases}, \\ {}^bT_p M^* &\cong \begin{cases} \langle \frac{dx_1}{x_1} |_p \rangle \oplus T_p Z^* & \text{if } p \in Z \\ T_p M^* & \text{if } p \notin Z \end{cases}, \end{aligned}$$

where x_1 is a defining function for Z^1 .

A generalization of this bundle can be constructed for more restrictive classes of vector fields if a defining function $x_1 \in \mathcal{C}^\infty(M)$ is fixed for $Z \subset M$:

Definition 2.5 The *b^k-tangent bundle* ${}^{b^k}TM$ is the vector bundle induced by the submodule generated locally by $\langle x_1^k \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle$. Thus, fiberwise this vector bundle has the form

$$\begin{aligned} {}^{b^k}T_p M &\cong \begin{cases} \langle x_1^k \partial_{x_1} |_p \rangle \oplus T_p Z & \text{if } p \in Z \\ T_p M & \text{if } p \notin Z \end{cases}, \\ {}^{b^k}T_p^* M &\cong \begin{cases} \langle \frac{dx_1}{x_1^k} |_p \rangle \oplus T_p^* Z & \text{if } p \in Z \\ T_p^* M & \text{if } p \notin Z \end{cases}, \end{aligned}$$

2.2 Relationship between TM and bTM

A natural question to ask ourselves is whether the *b*-tangent bundle is isomorphic to the usual tangent bundle for some choices of a manifold M and a hypersurface Z . There are cases in which both bundles are isomorphic. For instance:

Example 2.6 Let $\mathbb{T}^2$ denote the torus, considered as the quotient of $\mathbb{R}^2$ with respect to $\mathbb{Z}^2$, with coordinates (θ_1, θ_2) , and let $Z = \{\theta_1 = 0\} \cup \{\theta_1 = \frac{1}{2}\}$. If we take the vector fields $X_1 = \sin(2\pi\theta_1)\partial_{\theta_1}$ and $X_2 = \partial_{\theta_2}$, we see that ${}^b\mathfrak{X}(M) = \langle X_1, X_2 \rangle$, so ${}^bT\mathbb{T}^2$ is parallelizable, and thus it is isomorphic to $T\mathbb{T}^2$.

There are, however, plenty of cases in which the bundles are not isomorphic:

Proposition 2.7² Let M an orientable manifold and Z a hypersurface. Let G be the graph whose vertices are the connected components $M \setminus Z$, and such that the edges between two vertices C_1 and C_2 are the connected components of Z adjacent to both C_1 and C_2 . If bTM is parallelizable, then G is 2-colorable.

¹We say that $f \in \mathcal{C}^\infty(M)$ is a *defining function* of a hypersurface Z if $f^{-1}(0) = Z$ and f vanishes transversally in Z .

²This is a joint result with other members of the lab: Robert Cardona, Anastasia Matveeva, and Arnau Planas.

This result is analogous to the main theorem in [4], where the authors deduce the necessary condition for symplectic structures to exist in a b^k -manifold.

This result gives a necessary condition for the b -tangent bundle to be parallelizable, so there are cases in which we can prove that the bundles are not isomorphic. For instance, if we take $M = \mathbb{T}^2$ again and take $Z = \{\theta_1 = \frac{1}{2}\}$, we see that $\mathbb{T}^2 \setminus Z$ has one connected component, so the graph G has one node and one edge that connects it to itself, so bTM cannot be parallelizable. Thus, ${}^bT\mathbb{T}^2 \not\cong T\mathbb{T}^2$.

3 Symplectic geometry on b^k -manifolds

b^k -geometry provides a context in which we can define a class of structures that generalize the usual symplectic structure, staying within the setting of Poisson geometry. We can define this generalization in two equivalent ways. We begin by defining it as a concrete case of a Poisson structure.

Definition 3.1 A Poisson structure Π on a manifold M^{2n} is **b^k -Poisson** when:

1. Π^n vanishes at order k in a hypersurface $Z \subset M$. This means that, locally, if we have a coordinate system $(x_1, x_2, \dots, x_{2n})$ (with x_1 a defining function for Z) in an open U ,

$$\Pi^n|_U(x_1, x_2, \dots, x_{2n}) = x_1^k \partial_{x_1} \wedge \partial_{x_2} \wedge \dots \wedge \partial_{x_{2n}}.$$

2. $(\Pi^n)^{-1}(0) = Z$.

In order to tackle a different definition, as a generalization of symplectic structures, we need to define the differential for b^k -forms. Let ${}^b\Omega^m(M) = \Gamma\left(\Lambda^m\left({}^bT^*M\right)\right)$, the set of b^k - m -forms.

Lemma 3.2 For all $\eta \in {}^b\Omega^m(M)$ and for all f defining function of Z there exist $\alpha_0, \dots, \alpha_{k-1} \in \Omega^{m-1}(M)$ and $\beta \in \Omega^m(M)$ such that

$$\eta = \frac{df}{f^k} \wedge \left(\sum_{i=0}^{k-1} f^i \alpha_i \right) + \beta.$$

Using this we can define a differential $d : {}^b\Omega^m(M) \rightarrow {}^b\Omega^{m+1}(M)$ by imposing that the form $\frac{df}{f^k}$ is closed.

Definition 3.3 A 2-form $\omega \in {}^b\Omega^2(M)$ is **b^k -symplectic** if it is closed and non-degenerate.

Proposition 3.4 Definitions 3.1 and 3.3 are equivalent.

This proposition is proved in [1] using Weinstein's theorem³.

From here, we can derive results that reproduce the rigidity of symplectic geometry to the b^k -symplectic case:

Theorem 3.5 (Darboux Theorem for b^k -symplectic manifolds): For all $p \in Z$ there is a coordinate system centered at p , $(U, (x_1, y_1, \dots, x_n, y_n))$, such that x_1 is a defining function for $Z \cap U$ and

$$\omega|_U = \frac{dx_1}{x_1^k} \wedge dy_1 + dx_2 \wedge dy_2 + \dots + dx_n \wedge dy_n,$$

or, equivalently,

$$\Pi|_U = x_1^k \partial_{x_1} \wedge \partial_{y_1} + \partial_{x_2} \wedge \partial_{y_2} + \dots + \partial_{x_n} \wedge \partial_{y_n}.$$

³Weinstein's theorem asserts that, if (M, Π) is a Poisson manifold and $p \in M$, there exists a local chart $(U, (x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_m))$ such that

$$\Pi|_U = \sum_{i=1}^n \partial_{x_i} \wedge \partial_{y_i} + \sum_{1 \leq j < k \leq m} f_{jk}(z_1, \dots, z_m) \partial_{z_j} \wedge \partial_{z_k},$$

where f_{jk} vanish at 0. This can be seen as the best possible analogous to Darboux theorem for Poisson geometry.

As a particular case of a Poisson manifold, a b^k -symplectic manifold has a symplectic foliation. It is clear that the connected components of $M \setminus Z$ are symplectic leaves of dimension $2n$ of the foliation, but what happens inside the hypersurface is more particular to b^k -symplectic manifolds:

Theorem 3.6 *The symplectic foliation $\mathcal{F}$ associated to a b^k -symplectic form*

$$\omega = \frac{df}{f^k} \wedge \left(\sum_{i=0}^{k-1} f^i \alpha_i \right) + \beta$$

has maximal rank on Z , this means, $\text{rank}(\mathcal{F}) = 2n - 2$ on Z .

Moreover, the symplectic foliation on Z coincides with the foliation given by the 1-form $i_Z^ \alpha_0$, and the symplectic structure for each leaf $L \in \mathcal{F}$ is the one given by $i_L^* \beta$.*

4 Topological obstructions for b -symplectic manifolds

In this section we list some results which can be found in [1], where we can see more analogies of the restrictive behaviour of symplectic topology in the b^1 -symplectic case.

Theorem 4.1 (Mazzeo-Melrose): *The b -De Rham cohomology of (M, Z) satisfies that*

$${}^b H_{DR}^m(M) = H_{DR}^{m-1}(Z) \oplus H_{DR}^m(M)$$

This has two indirect consequences in the compact case:

Proposition 4.2 *For a compact b -symplectic manifold with $Z \neq \emptyset$, $H_{DR}^1(Z) \neq 0$. Therefore, as a consequence of 4.1, ${}^b H_{DR}^2(M) \neq 0$.*

Proposition 4.3 *For a compact b -symplectic manifold of dimension greater than 2 with $Z \neq \emptyset$, $H_{DR}^2(Z) \neq 0$. Therefore, as a consequence of 4.1, ${}^b H_{DR}^3(M) \neq 0$.*

Finally, we can see that, looking at b -symplectic manifolds as a particular case of Poisson manifolds, we are able to compute the Poisson cohomology (which is usually very complicated, in particular because the Poisson cohomology groups may have infinite rank):

Theorem 4.4 *Let (M, Z, Π) a b -Poisson manifold. Then, the Poisson cohomology given by Π is isomorphic to the b -De Rham cohomology:*

$$H_{\Pi}^{\bullet}(M) \cong {}^b H_{DR}^{\bullet}(M).$$

References

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