Problem List 6
Multivariate Calculus
Unit 4 - Integration in multiple variables

Lecturer: Prof. Sonja Hohloch, Exercises: Joaquim Brugués

1. Draw the following sets, and determine which ones are compact:
   (a) $A = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$.
   (b) $B = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$.
   (c) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4,\ z \in [-1, 1]\}$.
   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1,\ 0 \leq y \leq \frac{1}{x}\}$.
   (e) $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y = 5\}$.
   (f) $F = \{(x, y) \in \mathbb{R}^2 \mid x^2 + |y| = 5\}$.
   (g) $G = \{(x, y, z) \in \mathbb{R}^3 \mid x \in [-2, 1],\ y \in [0, 1],\ z = 5\}$.
   (h) $H = \{(0, 0, 0), (-1, 0, 1), (1, 0, 0)\}$.
   (i) $I = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid 1 \leq r \leq 2\}$.
   (j) $J = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4,\ (x-1)^2 + y^2 > 1\right\}$.

2. Argue which of the following functions are uniformly continuous in the given domains:
   (a) $f(x) = \frac{1}{x}$ in $D = [1, +\infty[$.
   (b) $g(x, y) = x^2y - xy^2$ in $E = [0, 2] \times [0, 2]$.
   (c) $h(x, y) = x^2 + y^2$ in $F = \mathbb{R}^2$.
   (d) $i(x) = \cos(x)$ in $H = \mathbb{R}$.
   (e) $j(x) = \cos(x^2)$ in $I = \mathbb{R}$.
   (f) $k(x) = \ln(x)$ in $J =]1, 2[$.

3. Let $F : A \subset \mathbb{R}^m \to \mathbb{R}^n$ a uniformly continuous function.
   (a) Prove that, if $(x_k)_k$ and $(y_k)_k$ are sequences contained in A such that $\lim_{k\to\infty}(x_k - y_k) = 0$, then $\lim_{k\to\infty}(F(x_k) - F(y_k)) = 0$.
   (b) Prove that if $(x_k)_k \subset A$ is a Cauchy sequence, then $(F(x_k))_k$ is also a Cauchy sequence.

4. Compute the integrals of the following functions in the given rectangles:
   (a) $f(x, y) = y^2$; in $|x| \leq 1$, $|y| \leq 2$.
   (b) $f(x, y) = x|y|$; in $0 \leq x \leq 2$, $-1 \leq y \leq 3$.
   (c) $f(x, y, z) = x^2 + y^2 + z^2$; in $-1 \leq x, y, z \leq 1$.
   (d) $f(x, y, z) = xyz$; in $0 \leq x, y, z \leq 1$.

5. Compute the integrals of the following functions in the given sets. It might be helpful to draw the set in each case.

   (a) $f(x, y) = x - y$; in $A = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1, x \geq 0, y \geq 0\}$.
   (b) $f(x, y) = x^2y^2$; in $B = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, |x| + |y| \leq 1\}$.
   (c) $f(x, y) = xy^2$; in $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.
   (d) $f(x, y) = xy$; where D is the region of the plane limited by the lines $\{x = 2\}$, $\{x = 4\}$, $\{y = x - 1\}$ and $\{y = 2x\}$.
   (e) $f(x, y) = x^2 - y$; where E is the region of the plane limited by the parabolas $\{y = x^2\}$ and $\{y = -x^2\}$ and the lines $\{x = -1\}$ and $\{x = 1\}$.
   (f) $f(x, y) = y$; in $F = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\ x^2 + y^2 \leq 4,\ y^2 \leq 3x\}$.
   (g) $f(x, y, z) = z$; in $G = \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1,\ |y| \leq 1,\ 0 \leq z \leq 2 - x^2 - y^2\}$.
   (h) $f(x, y, z) = \frac{1}{(1+x+y+z)^3}$; in $H = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0,\ x + y + z \leq 1\}$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ a continuous function. For each of the following integrals, draw the domain of integration and compute the expression of the integral changing the order of the variables.

   (a)
$$\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x, y)dxdy.$$

   (b)
$$\int_0^4 \int_{3x^2}^{12x} f(x, y)dydx.$$

7. Let $\varphi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ a change of coordinates, this means, a diffeomorphism. Let $f : K \subset V \subset \mathbb{R}^n \to \mathbb{R}$ a continuous function, and let us denote $g = f \circ \varphi$ and $K' = \varphi^{-1}(K)$. For each of the following changes of coordinates, find the expression of the integral

$$I = \int_K f(x_1, ..., x_n)dx_1 \cdots dx_n$$

   after changing the coordinates.

   (a) *(Polar coordinates):*

$$\begin{array}{rccc} \varphi: & [0, +\infty[\times]0, 2\pi[\subset \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (r, \theta) & \longmapsto & (r\cos\theta, r\sin\theta) \end{array}.$$

   (b) *(Cylindrical coordinates):*

$$\begin{array}{rccc} \varphi: & [0, +\infty[\times]0, 2\pi[\times\mathbb{R} \subset \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (r, \theta, z) & \longmapsto & (r\cos\theta, r\sin\theta, z) \end{array}.$$

   (c) *(Spherical coordinates):*

$$\begin{array}{rccc} \varphi: & [0, +\infty[\times]0, 2\pi[\times]0, \pi[\subset \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (r, \theta, \phi) & \longmapsto & (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \end{array}.$$

8. Compute the area of the following subsets of $\mathbb{R}^2$:

   (a) $A = \{(x, y) \mid x^2 \leq y \leq x\}$.
   (b) $B = \{(x, y) \mid x + y \geq 1,\ x^2 + y^2 \leq 1\}$.

(c) $C = \{(x, y) \mid (x + y)^2 + (2x - y + 1)^2 \leq 1\}$.

(d) D the ellipse with radii a and b, this means, $D = \left\{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$.

9. Compute the volume of the following subsets of $\mathbb{R}^3$:

   (a) A the ellipsoid with radii a, b and c, this means, $A = \left\{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$.

   (b) $B = \{(x, y, z) \mid x^2 + y^2 \leq 1,\ x + 2z \geq 0,\ y + 2z \leq 2\}$.

10. If $a, b, c \in [0, 1]$ are picked at random, what is the probability that the polynomial $p(x) = ax^2 + bx + c$ has real roots?

11. Compute
$$\iint_D (x^2 + y^2) dxdy,$$
where $D = \{(x, y) \in \mathbb{R}^2 \mid 2x \leq x^2 + y^2 \leq 4\}$.

12. Compute
$$\iint_D x dxdy,$$
where $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2x,\ y^2 \geq x\}$.

13. Compute
$$\iint_D \sqrt{1 + x^2} dxdy,$$
where D is the region of the plane delimited by the curves $y^2 - x^2 = 1$, $x = 0$ and $x = 1$.

14. Compute
$$\iint_D \sqrt{xy} dxdy,$$
where D is the region of $\mathbb{R}^2$ delimited by the curves $xy = 1$, $xy = 2$, $y^2 = x$ and $y^2 = 2x$.
(Hint: Try to use the new coordinates $u = xy, v = \frac{y^2}{x}$)

15. Use an appropriate change of coordinates to compute
$$\iint_T e^{x-y} dxdy,$$
where T is the triangle with vertices $(0, 0)$, $(1, 3)$ and $(2, 2)$.

16. Use polar coordinates to compute the integral
$$\iint_D \frac{x}{x^2 + y^2} dxdy,$$
where D is the region delimited by $y \leq x^2$, $x^2 + y^2 \leq 2$ and $y \geq 0$.

17. Let
$$I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r dr d\theta.$$
Write I in Cartesian coordinates and compute it.

18. Compute the area enclosed by a *simple rose petal*, described in polar coordinates by the equation $r = a \sin\theta$, where $a > 0$.

19. Let
$$I = \int_0^6 \int_{\frac{3}{2}x-3}^{x} e^{(3x-2y)^2}\,dydx.$$
   (a) Write the integral after changing the order of integration.
   (b) Find a suitable change of coordinates and compute the integral.

20. Let $0 < a < b$ and $0 < c < d$ and $D = \{(x, y) \in \mathbb{R}^2 \mid ax^2 \leq y \leq bx^2,\ cx \leq y \leq dx\}$. Compute
$$\iint_D \frac{1}{x} dxdy$$
using a suitable change of coordinates.

21. Compute
$$\iiint_A \sqrt{x^2 + y^2} dxdydz,$$
where $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1,\ x^2 + y^2 \leq z^2,\ z \geq 0\}$.

22. Compute the volume of A, the region limited by the surfaces $\{(x, y, z) \in \mathbb{R}^3 \mid z = 4 - y^2\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + 3y^2\}$.

23. Let $V = \{(x, y, z) \mid x, y, z \geq 0,\ z \leq x + y,\ x + y \leq 1\}$. Compute the integral
$$\iiint_V x dxdydz.$$

24. Let V be the region of $\mathbb{R}^3$ comprised between the spheres $S_1 : x^2 + y^2 + z^2 = 1$ and $S_2 : x^2 + y^2 + z^2 = 4$. Compute the integral
$$\iiint_V \sqrt{x^2 + y^2 + z^2} dxdydz.$$

25. Compute the length of the following curves using line integrals:
   (a) A circumference of radius R.
   (b) $\gamma(t) = (a \cos t, a \sin t, bt)$, $0 < t < 4\pi$, $a, b > 0$ (a helix).
   (c) The *astroid*, which satisfies the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

26. Compute the following line integrals:
   (a) $\displaystyle\int_C \sqrt{a^2 - y^2} dl$, where C is the curve given by $x^2 + y^2 = a^2$, $y > 0$.
   (b) $\displaystyle\oint_C (x^2 + y^2) dl$, where C is the circumference of center $(0, 0)$ and radius R.
   (c) $\displaystyle\int_C (xy + z^2) dl$, where C is the helix arc parametrized as $x = \cos t$, $y = \sin t$, $z = t$ and comprised between the points $(1, 0, 0)$ and $(-1, 0, \pi)$.

27. Compute the line integrals of the following vector fields:
   (a) $\displaystyle\int_C F \cdot dl$, where $F(x, y) = (x + y, y - x)$ and C is a part of the ellipse $x^2 + \frac{y^2}{4} = 1$ oriented from the point $(1, 0)$ to $(0, 2)$.

(b) $\displaystyle\int_L F \cdot dl$, where $F(x,y,z) = \left(\frac{1}{x-y}, \frac{1}{y-x}, z\right)$ and L is the segment from $(1,0,0)$ to $(2,1,2)$.

(c) $\displaystyle\int_C F \cdot dl$, where $F(x,y,z) = (2xy, 3z, 5yz)$ and C is the curve parametrized by $\gamma(t) = (t+1, t^3-1, t^2)$ from $(0,-2,1)$ to $(2,0,1)$.

(d) $\displaystyle\int_L F \cdot dl$, where $F(x,y,z) = (x, y, xz - y)$ and L is the segment from $(0,0,0)$ to $(1,2,4)$.

(e) $\displaystyle\int_C F \cdot dl$, where $F(x,y) = \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)$ and C is the circumference of center $(0,0)$ and radius $R > 0$ (taken in anticlockwise direction).

(f) $\displaystyle\int_C F \cdot dl$, where $F(x,y,z) = (3xy, -y^2, e^z)$ and C is the curve defined by the equations $z = 0, y = 2x^2$ from the point $(0,0,0)$ to the point $(1,2,0)$.

28. Let C be a curve on the surface of a sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq R^2\}$ and in the first octant, this means, such that $x, y, z > 0$. Prove that the line integral

$$\int_C \left(\frac{1}{yz}, \frac{1}{xz}, \frac{1}{xy}\right) \cdot dl = 0$$

for any choice of C and R.

Solutions

1. (a) A is compact.
   (b) B is not compact because it is not closed.
   (c) C is compact.
   (d) D is not compact because it is not bounded.
   (e) E is not compact because it is not bounded.
   (f) F is compact.
   (g) G is compact.
   (h) H is compact.
   (i) I is compact.
   (j) J is not compact because it is not closed.

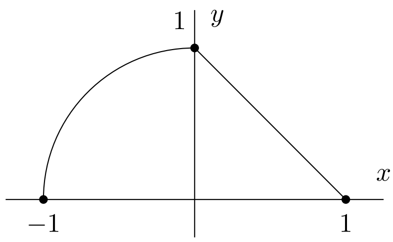
2. (a) f is uniformly continuous in D.
   (b) g is uniformly continuous in E.
   (c) h is not uniformly continuous in F.
   (d) i is uniformly continuous in H.
   (e) j is not uniformly continuous in I.
   (f) k is uniformly continuous in J.

3. Use the definition of uniform continuity, convergence of sequences and Cauchy sequence.

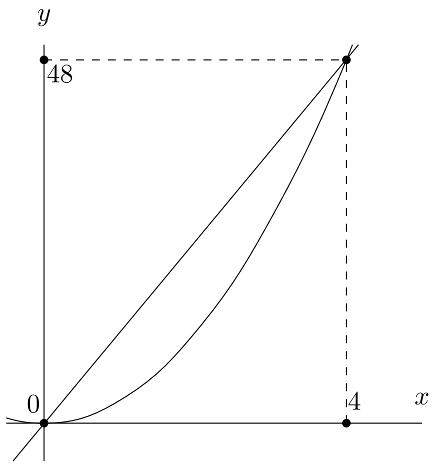
4. (a) $\frac{32}{3}$.
   (b) 10.
   (c) 8.
   (d) $\frac{1}{8}$.

5. (a) 0.
   (b) $\frac{1}{90}$.
   (c) $\frac{1}{15}$.
   (d) $\frac{317}{3}$.
   (e) $\frac{4}{5}$.
   (f) $\frac{19}{12}$.
   (g) $\frac{176}{45}$.
   (h) $\ln\left(\sqrt{2}\right) - \frac{5}{16}$.

6. (a) The domain has the following form:



$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x,y)dydx + \int_{0}^{1}\int_{0}^{1-x} f(x,y)dydx.$$



(b) The domain has the following form:

$$\int_0^{48}\int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x,y)dxdy.$$

7. (a)
$$I = \int_{K'} g(r,\theta) r dr d\theta.$$

(b)
$$I = \int_{K'} g(r,\theta,z) r dr d\theta dz.$$

(c)
$$I = \int_{K'} g(r,\theta,\phi) r^2 \sin\phi dr d\theta d\phi.$$

8. (a) $\frac{1}{6}$.
   (b) $\frac{\pi}{4} - \frac{1}{2}$.
   (c) $\frac{\pi}{3}$.
   (d) πab.

9. (a) $\frac{4}{3}\pi abc$.
   (b) π.

10. $\frac{5}{36} + \frac{1}{6}\ln 2 \approx 25.44\%$.

11.
$$\iint_D (x^2+y^2)dxdy = \frac{13}{2}\pi.$$

12.
$$\iint_D xdxdy = \frac{\pi}{2} - \frac{22}{15} \approx 0.1041.$$

13.
$$\iint_D \sqrt{1+x^2}dxdy = \frac{8}{3}.$$

14.
$$\iint_D \sqrt{xy}dxdy = \frac{2\ln 2}{9}\left(2\sqrt{2}-1\right).$$

15. The change of coordinates is $u = x - y$, $v = y - 3x$. The value of the integral is
$$\iint_T e^{x-y}dxdy = e^{-2}.$$

16. The integral expressed in polar coordinates is
$$\int_0^{\frac{\pi}{4}}\int_{\frac{\sin\theta}{\cos^2\theta}}^{\sqrt{2}} \cos\theta dr d\theta,$$
and its value is $1 - \frac{\ln 2}{2}$.

17. In Cartesian coordinates the integral has the form
$$\int_0^2\int_0^{\sqrt{1-(x-1)^2}} 1dydx,$$
and its value is $\frac{\pi}{2}$.

18. $\frac{a^2\pi}{4}$.

19. (a)
$$\int_0^6\int_y^{\frac{2}{3}y+2} e^{(3x-2y)^2}dxdy.$$

    (b) The change of coordinates is $u = 3x - 2y$, $v = x - y$. After changing variables the integral has the expression
$$\int_0^6\int_0^{\frac{u}{2}} e^{u^2}dvdu = \frac{e^{36}-1}{4}.$$

20. A possible change of coordinates is $u = \frac{y}{x^2}$, $v = \frac{y}{x}$. Then, the integral has the form
$$I = \int_a^b\int_c^d \frac{v}{u^2}dvdu = \frac{1}{2}(d^2-c^2)\left(\frac{1}{a}-\frac{1}{b}\right).$$

21. $\frac{\pi^2}{16} - \frac{\pi}{8}$.

22. 4π.

23. $\frac{1}{8}$.

24. 15π.

25. (a) $2\pi R$.
    (b) $4\pi\sqrt{a^2+b^2}$.
    (c) $6a$.

26. (a) $2a^2$.
    (b) $2\pi R^3$.
    (c) $\frac{\sqrt{2}}{3}\pi^3$.

27. (a) $\frac{3}{2} - \pi$.
    (b) 2.
    (c) $\frac{114}{35}$.
    (d) $\frac{23}{6}$.
    (e) -2π.
    (f) $-\frac{7}{6}$.

28. If we take a parametrization of C, $\gamma(t) = (x(t), y(t), z(t))$. As this curve lies on the surface of an sphere,

$$x^2(t) + y^2(t) + z^2(t) = R^2 \Rightarrow 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0.$$

Then,

$$\int_C \left(\frac{1}{yz}, \frac{1}{xz}, \frac{1}{xy}\right) \cdot dl = \int_{t_0}^{t_1} \left\langle \left(\frac{1}{yz}, \frac{1}{xz}, \frac{1}{xy}\right), (x', y', z') \right\rangle dt =$$
$$= \int_{t_0}^{t_1} \left(\frac{x'}{yz} + \frac{y'}{xz} + \frac{z'}{xy}\right) dt = \int_{t_0}^{t_1} \frac{xx' + yy' + zz'}{xyz} dt = 0.$$