Problem List 4
Multivariate Calculus
Unit 3 - Differentiability in multiple variables

Lecturer: Prof. Sonja Hohloch, Exercises: Joaquim Brugués

1. For each of the following functions, compute the maximal domain. Compute the partial derivatives at an arbitrary point of the domain.

   (a) $u(x,y) = \frac{x^2+y^2}{x^2-y^2}$.

   (b) $v(x,y,z) = \left(\sqrt{1-y^2-z^2}, \sqrt{1-z^2-x^2}, \sqrt{1-x^2-y^2}\right)$.

   (c) $v(x_1,...,x_n) = \displaystyle\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2$.

2. Let f be the function
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is it continuous at $(0,0)$?

   (b) Compute $\frac{\partial f}{\partial x}$. Is it continuous?

   (c) Compute $\frac{\partial f}{\partial y}$. Is it continuous?

   (d) Compute the directional derivative at 0 with respect to a generic vector $v = (a,b)$.

   (e) Is f differentiable at $(0,0)$?

3. Let f be the function
$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

   (a) Is it continuous at $(0,0)$?

   (b) Compute $\frac{\partial f}{\partial x}$. Is it continuous?

   (c) Compute $\frac{\partial f}{\partial y}$. Is it continuous?

   (d) Is f differentiable at $(0,0)$?

4. Is the function $f(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$ continuous at the point $(0,0)$? Do directional derivatives exist at $(0,0)$? Is it differentiable at $(0,0)$?

5. Study the continuity and differentiability at $(0,0)$ of the function
$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

6. Study the continuity and differentiability at $(0,0)$ of the function

$$f(x,y)=\begin{cases}\frac{(x+y)^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$$

7. Let $\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)$ the space of 2×2 matrices. We can identify this space with $\mathbb{R}^4$ with the mapping $F:\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)\to\mathbb{R}^4$ given by

$$F\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=(a,b,c,d)^T$$

Consider the map $\det:\mathcal{L}(\mathbb{R}^2,\mathbb{R}^2)\to\mathbb{R}$ that associates the determinant for each matrix.

(a) Show that det is differentiable.

(b) Compute the differential of det at the matrix $\mathrm{Id}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

(c) Compute the directional derivative of det at Id with direction $v=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

8. Compute the directional derivatives at the specified points and directions,

(a) $f(x,y)=\log\left(\sqrt{x^2+y^2}\right)$, $p=(1,0)$, $v=(2,1)$.

(b) $g(x,y)=e^x\cos(\pi y)$, $p=(0,-1)$, $v=(-1,2)$.

9. Compute all the points in $\mathbb{R}^2$ where the plane tangent to the graphic of the function $f(x,y)=(1-\sin x)y+y^3$ is parallel to the XY-plane. What is the equation for the tangent plane at those points?

10. Compute the partial derivatives of the following functions at an arbitrary point of their domain:

(a) $f(x,y,z)=y\arcsin(x\log y)+2^{\frac{x}{y}}$.

(b) $g(x,y,z)=xy^2+y^2z^3+xz^3$.

(c) $h(x,y,z)=x^{yz}$.

11. Use the chain rule to compute the partial derivatives of the following composite functions:

(a) $F=f\circ g$, where $f(x,y,z)=x^2y+y^2z-xyz$ and $g(u,v)=(u+v,u-v,u)$.

(b) $F=f\circ g$, where $f(x,y)=\frac{x+y}{1-xy}$ and $g(u,v)=(\tan u,\tan v)$.

12. Let $f:\mathbb{R}\to\mathbb{R}$ differentiable. Take $\varphi(x,y)=f(2x+3y)$. Show that this function satisfies the partial differential equation

$$3\frac{\partial\varphi}{\partial x}-2\frac{\partial\varphi}{\partial y}=0.$$

13. Let us consider the map giving the polar change of coordinates,

$$G:\begin{array}{ccc}]0,+\infty[\times]0,2\pi[& \longrightarrow & \mathbb{R}^2\\ (r,\theta) & \longmapsto & (r\cos\theta,r\sin\theta)\end{array}.$$

Let $f=(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$ a $\mathcal{C}^1$ function satisfying the Cauchy-Schwarz equations,

$$\begin{cases}\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y}\\ \frac{\partial f}{\partial y}=-\frac{\partial f_2}{\partial x}\end{cases}.$$

Show that the function $F=f\circ G$ satisfies the equations

$$\begin{cases}\frac{\partial F_1}{\partial r}=\frac{1}{r}\frac{\partial F_2}{\partial\theta}\\ \frac{\partial F_1}{\partial\theta}=-r\frac{\partial F_2}{\partial r}\end{cases}.$$

14. Let $F : \mathbb{R} \to \mathbb{R}^2$ the function given by $F(t) = (t^2, t^3)$. Is there any value $t_0 \in]0,1[$ such that $F(1) - F(0) = DF|_{t_0} \cdot (1-0)$? Does this contradict the Mean Value Theorem?

15. Let f the function given by

$$f(x,y) = \begin{cases} xy^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

Can we apply the Clairaut-Schwarz theorem to this function at the point $(0,0)$? Is it true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$?

16. Find the Taylor polynomial of degree 2 of the following functions:

(a) $f(x,y) = e^x \cos y$, around $p = (0,0)$.

(b) $g(x,y) = x^y$, around $p = (1,1)$.

(c) $h(x,y) = e^{x^2 + \sin y}$, around $p = (0,0)$.

(d) $i(x,y,z) = \frac{1}{1+x+y+z}$, around $p = (0,0,0)$.

17. Find and classify the critical points of the following functions:

(a) $f(x,y) = e^{1-x^2-y^2}$.

(b) $g(x,y) = x^2 + y^2 + 3xy$.

(c) $h(x,y) = y^2 - x^3$.

(d) $i(x,y) = (x-1)^2 + (x-y)^2$.

(e) $j(x,y) = \log(1 + x^2 + y^2)$.

(f) $k(x,y,z) = \cos(2x) \sin y + z^2$.

(g) $l(x,y,z) = x^2 - y^2 + (z-1)^4$.

(h) $m(x,y,z) = x + \frac{1+yz}{x}$.

(i) $n(x,y) = (x-y)^3 + (x+y)^2$.

18. Classify the critical points of the function

$$f(x,y) = \frac{1}{2}(x^2 + y^2) + \alpha xy$$

with respect to the parameter $\alpha \in \mathbb{R}$.

19. Consider the function $f(x,y) = (y - 3x^2)(y - x^2)$.

(a) Show that $(0,0)$ is its only critical point. Use the Hessian matrix to study its character.

(b) Consider the function $g = f \circ \gamma$, where $\gamma(t) = (at, bt)$ is any straight line through the origin. Show that g has a minimum at 0 independently of the chosen line.

(c) Prove that $(0,0)$ is not a minimum for f. *(Hint: Try to find a different curve through the origin such that the restriction of f to this curve does not have $(0,0)$ as a minimum.)*

Solutions

1. (a) $\mathrm{Dom}(u) = \{(x, y) \in \mathbb{R}^2 \mid |x| \neq |y|\}$.

$$\frac{\partial u}{\partial x} = \frac{-4xy^2}{(x^2 - y^2)^2}\ ,\ \frac{\partial u}{\partial y} = \frac{4yx^2}{(x^2 - y^2)^2}$$

(b) $\mathrm{Dom}(h) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, y^2 + z^2 \leq 1, x^2 + z^2 \leq 1\}$.

$$\frac{\partial h}{\partial x} = \left(0, \frac{-x}{\sqrt{1 - z^2 - x^2}}, \frac{-x}{\sqrt{1 - x^2 - y^2}}\right)$$
$$\frac{\partial h}{\partial y} = \left(\frac{-y}{\sqrt{1 - y^2 - z^2}}, 0, \frac{-y}{\sqrt{1 - x^2 - y^2}}\right)$$
$$\frac{\partial h}{\partial z} = \left(\frac{-z}{\sqrt{1 - y^2 - z^2}}, \frac{-z}{\sqrt{1 - z^2 - x^2}}, 0\right)$$

(c) $\mathrm{Dom}(v) = \mathbb{R}^n$.

$$\frac{\partial v}{\partial x_i} = 1 + 2x_i$$

2. (a) f is continuous at $(0, 0)$.

(b) $\frac{\partial f}{\partial x}$ is not continuous at the origin.

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2 + y^2)^2}$$

(c) $\frac{\partial f}{\partial y}$ is not continuous at the origin.

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

(d) The directional derivative is

$$Df|_{(0,0)} \cdot v = \frac{a^2 b}{a^2 + b^2}.$$

(e) f is not differentiable at $(0, 0)$.

3. (a) f is continuous at $(0, 0)$.

(b) $\frac{\partial f}{\partial x}$ is not continuous at the origin.

$$\frac{\partial f}{\partial x} = 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{x}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

(c) $\frac{\partial f}{\partial y}$ is not continuous at the origin.

$$\frac{\partial f}{\partial y} = 2y \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{y}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$$

(d) f is differentiable at $(0, 0)$, and its differential is $(0, 0)$.

4. The function is continuous at the origin. Directional derivatives do not exist, and thus it is not differentiable at the origin.

5. The function is continuous at the origin, but not differentiable.

6. The function is not continuous at the origin, and thus it is not differentiable.

7. (b) In vector form, $D\det|_{\mathrm{Id}} = (1,0,0,1)$.

   (c) Again using vector notation, $D\det|_{\mathrm{Id}} \cdot (0,-1,1,0)^T = 0$.

8. (a) $Df|_{(1,0)} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2.$

   (b) $Dg|_{(0,-1)} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1.$

9. The plane is tangent at all the points $\left(2\pi k + \frac{\pi}{2}, 0\right)$, where $k \in \mathbb{Z}$. The equation of the tangent plane at any of these points is $z = 0$.

10. (a)
$$\frac{\partial f}{\partial x}(x,y,z) = \frac{y \log y}{\sqrt{1-(x\log y)^2}} + \frac{\log 2}{y} 2^{\frac{x}{y}},$$
$$\frac{\partial f}{\partial y}(x,y,z) = \arcsin(x\log y) + \frac{x}{\sqrt{1-(x\log y)^2}} - \frac{(\log 2)x}{y^2} 2^{\frac{x}{y}}$$
$$\frac{\partial f}{\partial z}(x,y,z) = 0$$

    (b)
$$\frac{\partial g}{\partial x}(x,y,z) = y^2 + z^3 \ , \ \frac{\partial g}{\partial y}(x,y,z) = 2xy + 2yz^3 \ , \ \frac{\partial g}{\partial z}(x,y,z) = 3y^2z^2 + 3xz^2$$

    (c)
$$\frac{\partial h}{\partial x}(x,y,z) = yzx^{yz-1} \ , \ \frac{\partial h}{\partial y}(x,y,z) = z(\log x)x^{yz} \ , \ \frac{\partial h}{\partial z}(x,y,z) = y(\log x)x^{yz}$$

11. (a) $DF|_{(u,v)} = \left(3u^2 - 2uv + v^2, -u^2 + 2uv - 3v^2\right)$

    (b)
$$DF|_{(u,v)} = \frac{(1+\tan^2 u)(1+\tan^2 v)}{(1-\tan u \tan v)^2}(1,1).$$

    If we develop the expression and apply a trigonometric identity, we get that
$$DF|_{(u,v)} = \frac{1}{\cos^2(u+v)}(1,1).$$

12. The result follows from a direct application of the chain rule.

13. The result follows from applying the chain rule.

14. There exists no t_0 such that $F(1) - F(0) = DF|_{t_0} \cdot (1-0)$. This does not contradict the Mean Value Theorem because it only applies to functions whose image lies in $\mathbb{R}$.

15. The conditions of the Clairaut-Schwarz theorem are not satisfied because $\frac{\partial^2 f}{\partial x \partial y}$ is not continuous at $(0,0)$. However, it is true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

16. (a) $P_{f,2,(0,0)}(x,y) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2}$.
   (b) $P_{g,2,(1,1)}(x,y) = 1 + x + xy$.
   (c) $P_{h,2,(0,0)}(x,y) = 1 + y + x^2 + \frac{y^2}{2}$.
   (d) $P_{i,2,(0,0,0)}(x,y,z) = 1 - x - y - z + x^2 + y^2 + z^2 + 2(xy + yz + xz)$.

17. (a) There is only one critical point, $(0,0)$, a global maximum.
   (b) There is only one critical point, $(0,0)$, a saddle point.
   (c) There is only one critical point, $(0,0)$, a saddle point.
   (d) There is only one critical point, $(1,1)$, a global minimum.
   (e) There is only one critical point, $(0,0)$, a global minimum.
   (f) There are two infinite families of critical points, $p_k = \left(n\frac{\pi}{2}, n\pi + \frac{\pi}{2}, 0\right)$ and $q_n = \left(k\frac{\pi}{2} + \frac{\pi}{4}, k\pi, 0\right)$. All points are saddle points.
   (g) There is only one critical point, $(0,0,1)$, a saddle point.
   (h) There are two critical points, $(1,0,0)$ and $(-1,0,0)$. Both are saddle points.
   (i) There is only one critical point, $(0,0)$, a saddle point.

18. We can classify as follows:

   - If $|\alpha| > 1$, the only critical point, $(0,0)$, is a saddle.
   - If $\alpha \in]-1,1[$, the only critical point, $(0,0)$, is a minimum.
   - If $\alpha = \pm 1$, there are infinite critical points, given by $y = \mp x$, which are local minima.