Problem List 3
Multivariate Calculus
Unit 2 - Continuity, norms and metrics

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1. Let (U, d_U) and (V, d_V) metric spaces, and $f : U \to V$ a continuous map.
   (a) Prove that, if $C \subset V$ is a closed subset, then $f^{-1}(C) \subset U$ is closed.
   (b) Prove that, if $O \subset V$ is an open subset, then $f^{-1}(O) \subset U$ is open.
   (c) Use this to conclude that, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions, then
      i. The set $A = \{x \in \mathbb{R}^n \mid f(x) > g(x)\}$ is open.
      ii. The set $B = \{x \in \mathbb{R}^n \mid f(x) \geq g(x)\}$ is closed.

   (Hint: Prove one of the assertions, and use the fact that for any map $f : A \to B$ and any subset $C \subset B$, we have that $f^{-1}(C^c) = (f^{-1}(C))^c$)

2. Study the existence of the following limits:
   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^4}{\sqrt{x^2+y^2}}$
   (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3\sin xy}{xy}$
   (c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$
   (d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^6}$
   (e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5}{x^4+y^3}$
   (f) $\lim\limits_{(x,y,z)\to(0,0,0)} e^{\frac{-1}{x^2+y^2+z^2}}$

3. Let $A = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < x^2\}$ and

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A \\ 1 & \text{if } (x,y) \notin A \end{cases}.$$

   Prove that there exists a limit to $(0, 0)$ for any straight line going through the origin and it is equal to 1, but that the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

4. *(Uniform continuity):* Let (U, d_U) and (V, d_V) metric spaces, $A \subset U$ a subset and $f : A \to V$ a map. We say that f is *uniformly continuous* if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $d_U(x, y) < \delta \Rightarrow d_V(f(x), f(y))\ \forall x, y \in U$. Prove that
   (a) If $(x_n)_n \subset A$ is a Cauchy sequence, then $(f(x_n))_n$ is a Cauchy sequence.

(b) Use this result to show that the function $f :]0, 1[\to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

(c) Is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sin x$ uniformly continuous?

(d) Let $M > 0$. Is the function $h : [-M, M] \to \mathbb{R}$ given by $h(x) = x^2$ uniformly continuous?

(e) Is the function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ given by $\tilde{h}(x) = x^2$ uniformly continuous?

5. *(Equivalence of norms):* Let the usual norms in $\mathbb{R}^n$:

$$\|x\|_\Sigma = \sum_{i=1}^{n} |x_i| \quad \|x\|_E = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

In this problem we shall see that the three norms are equivalent between them.

(a) Show that $\|x\|_\infty \le \|x\|_E$ and $\|x\|_\infty \le \|x\|_\Sigma$ for any $x \in \mathbb{R}^n$.

(b) Show that $\|x\|_E \le \sqrt{n}\|x\|_\infty$.

(c) Show that $\|x\|_\Sigma \le n\|x\|_\infty$.

6. Consider the normed space $(\mathcal{C}([-1, 1], \mathbb{R}), \|\cdot\|_\infty)$ of continuous functions $f : [-1, 1] \to \mathbb{R}$, where the norm is

$$\|f\|_\infty = \sup_{x \in [-1,1]} |f(x)|.$$

Show that the following maps are continuous:

(a) $E_\alpha : \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$ given by $E_\alpha(f) = f(\alpha)$, where $\alpha \in [-1, 1]$.

(b) $I : \mathcal{C}([-1, 1], \mathbb{R}) \to \mathbb{R}$ given by $I(f) = \displaystyle\int_{-1}^{1} f(x)dx$.

(Hint: Prove that both are linear and bounded linear operators)