Problem List 2
Multivariate Calculus
Unit 1 - Sequences and Series

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1. Let $(x_n)_n$ a sequence of real numbers, and assume that it converges to a positive limit $l > 0$. Show that there exists some $N \in \mathbb{N}$ such that $x_n > 0 \ \forall n \geq N$.

2. Let $x_n = \frac{2n+3}{n+3}$ and $y_n = \frac{n+2}{4n+2}$. Are these sequences bounded? Are they increasing, strictly increasing, decreasing or strictly decreasing?

3. For each value of $\alpha > 0$, study whether the following sequence converges. In the case that it does, compute its limit.
$$x_n = \frac{\sin(\alpha n)}{1+\alpha^n}$$

4. Compute the following limits:
   (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+4}} + \cdots + \frac{1}{\sqrt{n^2+2n}}$
   (b) $\lim_{n\to\infty} \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$

5. Let $(x_n)_n$ and $(y_n)_n$ two sequences of real numbers such that $x_n \geq 0 \ \forall n \in \mathbb{N}$ and $\exists a, b \in \mathbb{R}$ such that $0 < a < y_n < b \ \forall n \in \mathbb{N}$.

   Let $z_n = (-1)^n x_n y_n$. Prove that $(z_n)_n$ converges if and only if $\lim_{n\to\infty} x_n = 0$.

6. Let $(x_n)_n$ the sequence of real numbers defined by
$$\begin{aligned} &x_0 = \tfrac{1}{2} \\ &7x_{n+1} = x_n^3 + 6 \end{aligned}$$
   Study its convergence. If it converges, compute its limit.

7. Let $(a_n)_n$ a sequence such that $a_n \geq 0$ for all n, and let us define the sequence $(b_n)_n$ such that $b_{2n} = a_n^2 - 6$ and $b_{2n+1} = a_n$ for all n.

   Show that if $(b_n)_n$ converges then $(a_n)_n$ also converges. Compute the limits.

8. *(Equivalent infinitesimals):* Let $(x_n)_n$ a sequence such that $\lim_{n\to\infty} x_n = 0$. Show that
   (a) $\lim_{n\to\infty} \frac{\sin(x_n)}{x_n} = 1$.
   (b) $\lim_{n\to\infty} \frac{\tan(x_n)}{x_n} = 1$.
   (c) $\lim_{n\to\infty} \frac{1-\cos(x_n)}{x_n^2} = \frac{1}{2}$.

(Hint: For the first two limits, use the fact that, if $0 < x < \frac{\pi}{2}$, then $\sin x \leq x \leq \tan x$, and find a way to apply the sandwich convergence criterion)

9. Let $(a_k)_k$ a sequence such that $a_k \geq 0$. Prove that

$$\sum_{k \geq k_0} a_k \text{ convergent} \Rightarrow \sum_{k \geq k_0} a_k^2 \text{ convergent.}$$

Show that this assertion is false if $(a_k)_k$ is not always positive.

10. Study the convergence of the following series:

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k - \sqrt{k}}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k}$.

(c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$.

(d) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\log k}{k}$.

(e) $\displaystyle\sum_{k=0}^{\infty} \frac{\arctan k}{k^3 + 1}$.

(f) $\displaystyle\sum_{k=1}^{\infty} k \left(\frac{3}{4}\right)^k$.

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{e^k}{k!}$.

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos^2 k}{\sqrt{k}}$.

(i) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log k}$.

11. Let $z > 0$, and take $A_n = \displaystyle\sum_{k=0}^{n} z^k$ and $B_n = \displaystyle\sum_{k=1}^{n} k z^{k-1}$.

(a) Prove that $B_{n+1} = zB_n + A_n \ \forall n \geq 0$.

(b) For which values of z does $(B_n)_n$ converge? Compute the limit if it exists.

12. Study the convergence of $\displaystyle\sum_{k=1}^{\infty} k^2 \alpha^k$ with respect to the parameter α.

13. Study the convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(\tan \alpha)^k}{k^3}$ with respect to the parameter α.

14. Let $(a_k)_k$ and $(b_k)_k$ sequences such that $a_k, b_k \geq 0 \ \forall k \geq 0$, and assume that $\displaystyle\sum_{k=0}^{\infty} a_k$ is convergent.

(a) Prove that if $(b_k)_k$ is bounded then $\sum_{k=0}^{\infty} a_k b_k$ is convergent.

(b) Prove that if $\sum_{k=0}^{\infty} b_k$ is convergent then $\sum_{k=0}^{\infty} \sqrt{a_k b_k}$ is convergent.

(c) In particular, show that $\sum_{k=0}^{\infty} \frac{\sqrt{a_k}}{k}$ converges.

15. *(Banach spaces)*: Let $(E, \|\cdot\|)$ a normed vector space. Prove that $(E, \|\cdot\|)$ is a Banach space if and only if for any sequence $(a_k)_k \subset E$ we have that $\sum_{k=0}^{\infty} a_k$ absolutely convergent $\Rightarrow \sum_{k=0}^{\infty} a_k$ convergent.

16. Compute the convergence radius for the following power series

(a) $\sum_{k=0}^{\infty} x^{5k}$

(b) $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k^3}$

(c) $\sum_{k=0}^{\infty} x^{k!}$

(d) $\sum_{k=0}^{\infty} r^{k^2} x^k$, where $|r| < 1$.