Problem List 1
Multivariate Calculus
Unit 1 - Distances, norms and topology

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1. Let A, B two nonempty subsets of $\mathbb{R}^n$, and d the Euclidean distance. Define $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.
   (a) Is it true that $d(A, B) = 0 \Leftrightarrow A \cap B \neq \emptyset$?
   (b) If $x \in \mathbb{R}^n$, let $d(x, A) = d(\{x\}, A)$. Show that $|d(x, A) - d(y, A)| \leq d(x, y)$ $\forall x, y \in \mathbb{R}^n$.
2. Let A a nonempty subset of $\mathbb{R}^n$, and consider $D_A = \{d(x, y) \mid x, y \in A\} \subset \mathbb{R}$. Prove that A is a bounded set if and only if D_A is bounded.
3. Take the *discrete metric* in $\mathbb{R}^n$, so that
$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$
   Show that for any point $x \in \mathbb{R}^n$, the set $\{x\}$ is open. Show that any subset $A \subset \mathbb{R}^n$ is open with respect to the discrete metric.
4. Let $(\mathbb{R}^n, d)$ the Euclidian space. Show that for any point $x \in \mathbb{R}^n$, the set $\{x\}$ is closed.
5. Compute the interior and closure of the following sets:
   (a) $A = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$.
   (b) $B = \{(x, y) \in \mathbb{R}^2 \mid x = \lambda y\}$.
   (c) $C = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$.
   (d) $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, y = 0\}$.
   (e) $E = A \backslash \{(0, 1)\}$.
   (f) $F = B \cup \{(0, -1)\}$.
   (g) $G = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, x \neq 0\}$.
   (h) $H = \mathbb{Q} \subset \mathbb{R}$.
   (i) $I = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \mid A^T = A\}$, the symmetric matrices.
   (j) $J = \{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \mid \det(A) \neq 0\}$, the regular matrices.
   (k) $K = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2, y = 3, z \in]-1, 1[\}$.
   (l) $L = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y = 5\}$.
   (m) $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + |y| = 5\}$.
   (n) $N = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1, |z| < 1\}$.
6. *(Characterizations)* Let A a subset of a metric space.

(a) Show that $x \in \operatorname{Int}(A)$ if and only if for all sequences $(x_n)_n$ converging to x $\exists N \in \mathbb{N}$ such that $x_n \in A$ $\forall n \geq N$.

(b) Show that $x \in \overline{A}$ (the closure) if and only if there exists a sequence $(x_n)_n$ contained in A and such that $x_n \xrightarrow[n\to\infty]{} x$.

7. Prove that, for any A subset of a metric space, $\operatorname{Int}(A)$ is the largest open set contained in A. This means, for any $\mathcal{O} \subset A$ open, we have that $\mathcal{O} \subset \operatorname{Int}(A)$.

8. Prove that, for any A subset of a metric space, $\overline{A}$ is the smallest closed set containing A. This means, for any $C \supset A$ closed, we have that $\overline{A} \subset C$.

9. Let $A \subset \mathbb{R}^n$ nonempty and d the Euclidian metric. Prove that if A is complete, then it is closed.

10. *(Norms for continuous functions)* Consider $\mathcal{C}[0,1]$ the set of continuous functions $f : [0,1] \to \mathbb{R}$.

    (a) Show that $\|f\|_\infty := \sup\limits_{x\in[0,1]} |f(x)|$ is well defined and it defines a norm on the vector space $\mathcal{C}[0,1]$ *(Hint: Use the Weierstrass theorem)*.

    (b) Show that the map $\|\cdot\|_I : \mathcal{C}[0,1] \to \mathbb{R}$ given by
    $$\|f\|_I = \int_0^1 |f(x)|dx$$
    is a norm.

    (c) Compute $\|f\|_\infty$ and $\|f\|_I$ for the function $f(x) = -x^2 + 4x + \frac{4}{3}$.

    (d) Compute $\|g\|_\infty$ and $\|g\|_I$ for the function $g(x) = \sin(\pi x)$.

    (e) Show that for all $f \in \mathcal{C}[0,1]$ we have $\|f\|_I \leq \|f\|_\infty$.

    (f) Show that if a sequence $(f_n)_n$ converges to f in the normed space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, then it also converges to the same limit f in the normed space $(\mathcal{C}[0,1], \|\cdot\|_I)$.

    (g) Is there any constant $C > 0$ such that we have $\|f\|_\infty \leq C\|f\|_I$ for all $f \in \mathcal{C}[0,1]$?

11. *(Completion of the rational numbers)* Consider the rational numbers as a normed space $(\mathbb{Q}, |\cdot|)$. We know that there are Cauchy sequences within this space that are not convergent, so it is not complete. We may, however, perform a construction based on this space in a way that *completes* it.

    (a) Show that if $(a_n)_n$ and $(b_n)_n$ are Cauchy sequences of rational numbers then $(a_n + b_n)_n$ and $(a_n b_n)_n$ are also sequences of rational numbers.

    (b) Let C denote the space of Cauchy sequences. Show that the relation $(a_n)_n \sim (b_n)_n \Leftrightarrow (a_n - b_n) \xrightarrow[n\to\infty]{} 0$ is an equivalence, this means,

    i. It is reflexive, so $(a_n)_n \sim (a_n)_n$.

    ii. It is symmetric, so $(a_n)_n \sim (b_n)_n \Leftrightarrow (b_n)_n \sim (a_n)_n$.

    iii. It is transitive, so if $(a_n)_n \sim (b_n)_n$ and $(b_n)_n \sim (c_n)_n$ then $(a_n)_n \sim (c_n)_n$.

    Also, prove that this relation respects the field operations, this means, if $(a_n)_n \sim (b_n)_n$ and $(c_n)_n \sim (d_n)_n$, then $(a_n + c_n)_n \sim (b_n + d_n)_n$.

    (c) Let $\mathcal{R} := C/\sim$. Prove that it is a field with respect to the addition and the product of sequences.

    (d) Prove that there is an injection $\mathbb{Q} \hookrightarrow \mathcal{R}$.

    (e) Let $a : \mathbb{N} \to \mathcal{R}$ a sequence, and denote it as $(a_{m,n})_{m,n}$. This means that $(a_{m,n})_n$ is an equivalence class of Cauchy sequences for each fixed m. We say that it is Cauchy if, $\forall\varepsilon > 0$ there is some N such that $|a_{m,n} - a_{l,n}| < \varepsilon$ $\forall m, l, n \geq N$.
    Prove that all Cauchy sequences in $\mathcal{R}$ converge.

Solutions

1. (a) It is false. For instance, in $\mathbb{R}$ we have that $d\left(]-1,0[,]0,1[\right)=0$ but $]-1,0[\bigcap]0,1[=\emptyset$.

   (b) If $x\in A$ or $y\in A$, the assertion is true because of the definition of the infimum. Otherwise, construct the sequences $(x_n)_n$ and $(y_n)_n$ in such a way that

$$x_n\in A\bigcap B\left(x,d(x,A)+\frac{1}{n}\right)\ ,\ y_n\in A\bigcap B\left(x,d(y,A)+\frac{1}{n}\right).$$

   This means that

$$d(x,A)=\lim_{n\to\infty}d(x,x_n)\ ,\ d(y,A)=\lim_{n\to\infty}d(y,y_n).$$

   Then, for all n we have that

$$d(x,A)-d(y,y_n)\leq d(x,y_n)-d(y,y_n)\leq d(x,y)+d(y,y_n)-d(y,y_n)=d(x,y),$$

   and

$$\lim_{n\to\infty}d(x,A)-d(y,y_n)=d(x,A)-d(y,A).$$

   Conversely, for all n we have that

$$d(y,A)-d(x,x_n)\leq d(y,x_n)-d(x,x_n)\leq d(y,x)+d(x,x_n)-d(x,x_n)=d(x,y),$$

   and

$$\lim_{n\to\infty}d(y,A)-d(x,x_n)=d(y,A)-d(x,A).$$

2. ($\Rightarrow$) If A is bounded, then there exist $x\in\mathbb{R}^n$, $R>0$ such that $A\subset B(x,R)$. Thus, by definition $D_A\subset D_{B(x,R)}$. Moreover, by construction $D_{B(x,R)}\subseteq[0,2R[$, which means that $D_A\subset[0,2R[$, so D_A is bounded.

   ($\Leftarrow$) If D_A is bounded, then there exists some $M>\sup D_A$. Let us take some $x\in A$. Then, we claim that $A\subset B(x,M)$.

   Let us argue by contradiction. Assuming that $A\not\subset B(x,M)$, we must have some $y\in A$ with $y\notin B(x,M)$, so that $d(x,y)\geq M$. However, M is strictly greater than the supremum of D_A, so we reach a contradiction.

   Therefore $A\subset B(x,M)$, so A is bounded.

3. By construction, $\{x\}=B_\rho\left(x,\frac{1}{2}\right)$, so it is open.

   For any set $A\subset\mathbb{R}^n$ we have that

$$A=\bigcup_{a\in A}\{a\},$$

   which as we just saw are open sets. Thus, A is open as well.

4. Let $x\in\mathbb{R}^n$. Then,

$$\{x\}^c=\{y\in\mathbb{R}^n\mid x\neq y\}=\bigcup_{y\neq x}B(y,d(x,y)),$$

   which is open.

5. (a) $\mathrm{Int}(A)=A$, and $\overline{A}=\{(x,y)\in\mathbb{R}^2\mid y\geq 0\}$.

   (b) $\mathrm{Int}(B)=\emptyset$, and $\overline{B}=B$.

   (c) $\mathrm{Int}(C)=\emptyset$, and $\overline{C}=C$.

   (d) $\mathrm{Int}(D)=\emptyset$, and $\overline{D}=\{(x,y)\in\mathbb{R}^2\mid 0\leq x\leq 1, y=0\}$.

(e) $\text{Int}(E) = E$, and $\overline{E} = \overline{A}$.

(f) $\text{Int}(F) = \emptyset$, and $\overline{F} = F$.

(g) $\text{Int}(G) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, x \neq 0\}$, and $\overline{G} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

(h) $\text{Int}(H) = \emptyset$, and $\overline{H} = \mathbb{R}$.

(i) $\text{Int}(I) = \emptyset$, and $\overline{I} = I$.

(j) $\text{Int}(J) = J$, and $\overline{J} = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

(k) $\text{Int}(K) = \emptyset$, and $\overline{K} = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2, y = 3, z \in [-1, 1]\}$.

(l) $\text{Int}(L) = \emptyset$, and $\overline{L} = L$.

(m) $\text{Int}(M) = \emptyset$, and $\overline{M} = M$.

(n) $\text{Int}(N) = N$, and $\overline{N} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, |z| \leq 1\}$.

6. (a) ($\Rightarrow$) Let $(x_n)_n$ a sequence converging to x. As $x \in \text{Int}(A)$, we know that $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset \text{Int}(A)$. By the definition of convergence, there exists $N \in \mathbb{N}$ with $x_n \in B(x, \varepsilon)$ $\forall n \geq N$, so $x_n \in \text{Int}(A)$ $\forall n \geq N$.

   ($\Leftarrow$) Let us prove the converse, so if $x \notin \text{Int}(A)$ then there exists a sequence satisfying the contrary to our statement. If $x \notin \text{Int}(A)$, then $\forall \varepsilon > 0$ we have that

   $$B(x, \varepsilon) \bigcap A^c \neq \emptyset.$$

   Thus, for each $n \in \mathbb{N}$ we can pick

   $$x_n \in B\left(x, \frac{1}{n}\right) \bigcap A^c.$$

   This means that we can construct a sequence $(x_n)_n$ such that $x_n \notin A$ $\forall n$.

   (b) ($\Rightarrow$) As in the last proof, let us construct a sequence by

   $$x_n \in B\left(x, \frac{1}{n}\right) \bigcap A \neq \emptyset.$$

   By construction, $(x_n)_n \subset A$ and $\lim\limits_{n\to\infty} x_n = x$. ($\Leftarrow$) Let $(x_n)_n$ a sequence converging to x and contained in A. Let $r > 0$. By the definition of convergence, there exists some $N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n \in B(x, r)$. Thus,

   $$x_n \in A \bigcap B(x, r),$$

   and therefore $A \bigcap B(x, r) \neq \emptyset$.

7. Let $\mathcal{O} \subset A$ an open subset. Then, for all $x \in \mathcal{O}$ there exists some $r > 0$ such that $B(x, r) \subset \mathcal{O} \subset A$, which implies that $x \in \text{Int}(A)$.

8. Let $C \supset A$ a closed set. Let $x \in \overline{A}$. Then, for all $r > 0$ we have that $B(x, r) \cap A \neq \emptyset$, so $B(x, r) \cap C \neq \emptyset$. This means that $x \in \overline{C}$, but, as C is closed, this means that $x \in C$.

9. Let $x \in \overline{A}$. This means that $\exists (x_n)_n \subset A$ such that $\lim\limits_{n\to\infty} x_n = x$. As $(x_n)_n$ is convergent in $\mathbb{R}^n$, it is a Cauchy sequence. Moreover, as $(x_n)_n \subset A$ is a Cauchy sequence and A is complete, it has to be convergent in A. Thus, $x \in A$.

   In conclusion, $\overline{A} = A$, so A is closed.

10. (a) By the Weierstrass Theorem, if $f : [0,1] \to \mathbb{R}$ is continuous then it must attain a maximum M_1 and a minimum M_2 within its domain. Thus, the norm is always well defined, and $\|f\|_\infty = \max\{|M_1|, |M_2|\}$.

Regarding the properties of the norm,

i. $\|f\|_\infty = 0$ if and only if $|f(x)| \leq 0$ for all $x \in [0,1]$, so $f(x) = 0$ for all x, and thus $f \equiv 0$.

ii. $\|\lambda f\|_\infty = \sup_{x\in[0,1]} |\lambda f(x)| = |\lambda| \sup_{x\in[0,1]} |f(x)| = |\lambda| \|f\|_\infty$.

iii.

$$\|f+g\|_\infty = \sup_{x\in[0,1]} |f(x)+g(x)| \leq \sup_{x\in[0,1]} |f(x)| + \sup_{y\in[0,1]} |g(y)| = \|f\|_\infty + \|g\|_\infty.$$

(b) Again, let us go over the properties of a norm,

i. $\|f\|_I = 0$ implies that $f \equiv 0$. Otherwise, if $\exists x \in [0,1]$ such that $|f(x)| > 0$, then there exists $\varepsilon > 0$ such that $|f(y)| > 0$ $\forall y \in]x-\varepsilon, x+\varepsilon[$, so

$$\int_0^1 |f(x)|dx \geq \int_{x-\varepsilon}^{x+\varepsilon} |f(x)|dx > 0.$$

Moreover, $\|0\|_I = 0$.

ii.

$$\|\lambda f\|_I = \int_0^1 |\lambda f(x)|dx = |\lambda| \int_0^1 |f(x)|dx = |\lambda| \|f\|_I.$$

iii.

$$\|f+g\|_I = \int_0^1 |f(x)+g(x)|dx \leq \int_0^1 (|f(x)| + |g(x)|)\, dx =$$

$$= \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx = \|f\|_I + \|g\|_I.$$

(c) f is a convex parabolla with the vertex at 2, so there is the global maximum of the function. Moreover, $f(0), f(1) > 0$, so we deduce that there cannot be any root of f within $[0,1]$, so therefore

$$\|f\|_\infty = |f(1)| = \frac{13}{3}.$$

On the other hand,

$$\|f\|_I = \int_0^1 |f(x)|dx = \int_0^1 -x^2 + 4x + \frac{4}{3} dx = \left[-\frac{x^3}{3} + 2x^2 + \frac{4}{3}x\right]_0^1 = 3.$$

(d) $\|g\|_\infty = 1$,

$$\|g\|_I = \int_0^1 \sin(\pi x) dx = \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1 = \frac{2}{\pi}.$$

(e)

$$\|f\|_I = \int_0^1 |f(x)|dx \leq \int_0^1 \|f\|_\infty dx = (1-0)\|f\|_\infty = \|f\|_\infty$$