A Floer complex for b-symplectic manifolds

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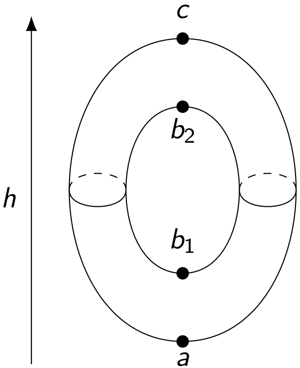
Contents

Motivation (I)

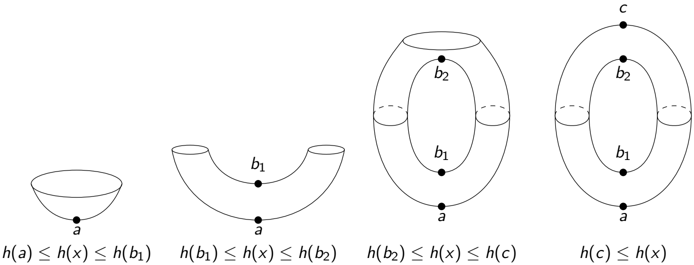
Torus embedded in $\mathbb{R}^3$:



Critical points in the torus

Motivation (II)

Region under the plane $h(x) = K$:



Formation of the torus

Morse homology

Let M a smooth manifold and $f : M \to \mathbb{R}$ a Morse function.

- Generators (over $\mathbb{Z}_2$): Critical points of f.
- Classified by the index.
- Points are related by the flow of $-\mathrm{grad}_g f$ (g Riemannian metric).
- Boundary operator:

$$\partial_k p = \sum_{q \in \mathrm{Crit}_{k-1}(f)} n_g(p,q) q,$$

  with $n_g(p,q)$ the number of trajectories between p and q.
- Invariance: $HM_\bullet(M)$ does not depend on f or g.
- Morse homology is isomorphic to simplicial homology.

Results

The homology $HM_\bullet(M)$ is a **topological invariant**.

Theorem (Morse inequalities)

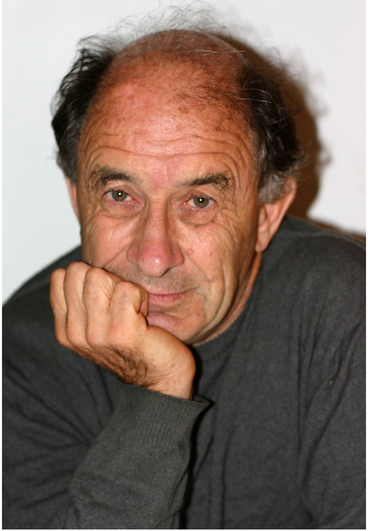
Let $f : M \to \mathbb{R}$ a Morse function, c_k the number of critical points of index k, and β_k the k-th Betti number of M. Then,

$$c_k \geq \beta_k.$$

In particular,

$$\#\mathrm{Crit}(f) \geq \sum_{k=0}^{n} \beta_k.$$

Arnold conjecture



Vladimir Arnold

Arnold conjecture

Theorem (Arnold 1963)

Consider $H_t : S^1 \times M \to \mathbb{R}$ a non-degenerate time-dependent and 1-periodic Hamiltonian. Then,

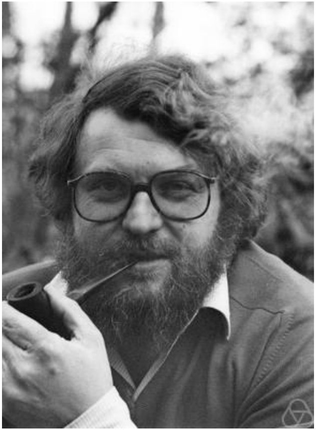
$$\#\{\textit{1-periodic orbits of } X_{H_t}\} \geq \sum_{k=0}^{2n} \beta_k.$$

Remark

In the case of an autonomous Hamiltonian H, this is a consequence of the Morse inequalities.

Solution: Floer homology (1988)

Andreas Floer



Eduard Zehnder

The Floer complex

We use the same idea as in the Morse complex

- Our domain is $\mathcal{L}M := \{x \in \mathcal{C}^\infty(S^1, M) \mid \text{contractible}\}$.
- Our function is the action functional $\mathcal{A}_H : \mathcal{L}M \longrightarrow \mathbb{R}$,

$$\mathcal{A}_H(x) := \int_0^1 H_t(x(t))dt - \int_{D^2} w^*\omega.$$

Remark

The critical points of $\mathcal{A}_H$ are precisely the 1-periodic orbits of X_H.

The critical points are classified by the Conley-Zehnder index μ_{CZ}.

Almost complex structures

Definition

An **almost complex structure** J on M is a section of $TM \otimes T^*M$ such that

$$J^2 = -\mathrm{Id}.$$

It is **callibrated by** ω if

- $\omega(JX, JY) = \omega(X, Y)\ \forall X, Y \in \mathfrak{X}(M)$.
- $\omega(X, JX) > 0\ \forall X \in \mathfrak{X}(M)$.

This induces a Riemannian metric on M.

The Floer equation

The (negative) gradient lines of $\mathcal{A}_H$ on (M, ω, J) are $u : \mathbb{R} \times \mathbb{S}^1 \to M$ such that satisfy the **Floer equation**:

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \mathrm{grad}_u H_t = 0$$

This is a generalization of **pseudoholomorphic curves** (studied by Gromov, 1985).

To connect critical points, we must impose the condition that

$$E(u) := \iint_{\mathbb{R} \times S^1} \left| \frac{\partial u}{\partial s} \right|^2 < +\infty$$

The Floer complex

The moduli space of solutions of the Floer equation $\mathcal{M}(x, y)$ is a compact manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y) - 1$.
This way we can define the boundary map of the complex, $\partial_k : CF_k(M, \omega, H, J) \to CF_{k-1}(M, \omega, H, J)$ over generators as

$$\partial_k(x) = \sum_{\mu_{CZ}(y)=k-1} \#\mathcal{M}(x, y)y$$

We define the Floer homology as

$$HF_k(M) := \frac{\mathrm{ker}\partial_\mathrm{k}}{\mathrm{im}\partial_{\mathrm{k+1}}}$$

Results

Theorem

The Floer homology $HF_\bullet(M)$ is a **topological invariant***: It does not depend on ω, H_t or J.*

Theorem

The Floer homology of a manifold is isomorphic to the Morse homology (with an index shifting):

$$HF_\bullet(M) \cong HM_{\bullet+n}(M).$$

This proves the Arnold conjecture for a wide class of manifolds.

b-symplectic manifolds

Definition

A b-symplectic manifold is a Poisson manifold (M^{2n}, π) such that $\pi^n \pitchfork 0$. If $Z := (\pi^n)^{-1}(0)$, then the symplectic foliation consists of

- The connected components of $M \backslash Z$ (dimension $2n$).
- Z is foliated by dimension $2n-2$ symplectic leaves.

In particular, Z has a cosymplectic structure.

Normal vector field

Definition

Consider (M, Z, ω) a b-symplectic manifold with Z orientable. Then there exists a **normal b-vector field** X^σ satisfying that

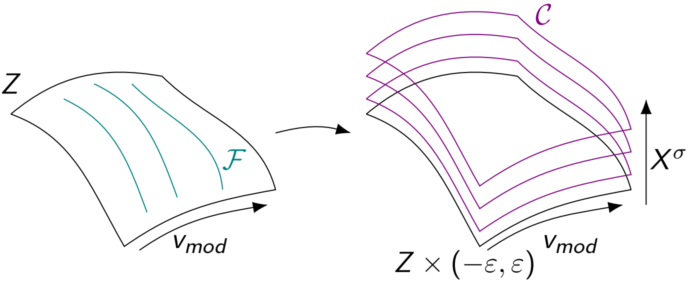
1. It is *symplectic*, this means, $\mathcal{L}_{X^\sigma}\omega = 0$.
2. It is transversal to Z in the sense that $X^\sigma_p \notin TZ$ for all $p \in Z$.

In local coordinates $(z, \theta, x_2, y_2, ..., x_n, y_n)$ the normal vector field has the expression $X^\sigma = z\frac{\partial}{\partial z}$.

In this case, X^σ is conjugate with the *modular vector field*, this means, $\omega(X^\sigma, v_{mod}) = 1$.

Cosymplectic foliation

This choice induces a foliation of $(-\varepsilon, \varepsilon) \times Z$ by cosymplectic hypersurfaces:



Admissible Hamiltonians (I)

Our purpose is to construct a Floer complex for $M \backslash Z$ for Hamiltonians with a relation with X^σ.

Definition

A Hamiltonian $H : S^1 \times M \to \mathbb{R}$ is **admissible** if

- It is invariant with respect to the modular vector field: $\mathcal{L}_{v_{mod}} H_t = 0$.
- It grows linearly in the normal direction: $\mathcal{L}_{X^\sigma} H_t = k(t)$ for some $k : S^1 \to \mathbb{R}$.
- Its Hamiltonian vector field exhibits no 1-periodic orbits in a tubular neighbourhood small enough around Z.

Locally, the Hamiltonian has the expression $H_t = k(t) \log |z| + h_t(x)$, where h is a function defined on the leaves of the symplectic foliation.

Admissible Hamiltonians (II)

Lemma

A sufficient condition to guarantee that X_H has no periodic orbits near Z is that

$$\int_{S^1} k(t)dt \in (0, T),$$

where T is the modular weight of Z.

Idea of proof: In local coordinates, $X_H = k(t)v_{mod} + X_h$, and each leaf of $\mathcal{C}$ has the form

$$\sigma = \frac{\mathcal{L} \times [0, T]}{(x, 0) \sim (f(x), T)}$$

The Floer equation

Solving the Floer equation in this context,

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \mathrm{grad}_u H_t = 0$$

We get that in a neighbourhood of Z, if we take $f(p) = -\log|z|$, then

$$\Delta(f \circ u) = 0$$

Results

- Using these conditions we can construct a Floer complex as in the standard case, connecting the 1-periodic orbits of X_H on $M \backslash Z$ using the Floer equation.
- The compactness of $\mathcal{M}(x, y)$ can be derived from a modified Maximum Principle.
- There exists a Floer homology, depending exclusively on the topology of (M, Z).

We conjecture that the homology is isomorphic to the relative homology

$$^{b}HF_{\bullet}(M, Z) \cong H_{\bullet+n}(M, Z)$$