Semilocal dynamics of b-symplectic manifolds

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August 16, 2022

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Symplectic Dynamics Beyond Periodic Orbits

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b-manifolds

Let M be a compact manifold, $Z \subset M$ an embedded hypersurface.

The set ${}^b\mathfrak{X}(M, Z) := \{X \in \mathfrak{X}(M) \mid X_p \in T_pZ \; \forall p \in Z\}$ is a **projective module**.
[Serre-Swan Theorem] $\Rightarrow$ there exists a vector bundle ${}^bTM \to M$ such that $\Gamma({}^bTM) = {}^b\mathfrak{X}(M, Z)$, the *b*-**tangent bundle**.

b-cotangent bundle

The *b*-**cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of ${}^b\Omega^k(M) := \Lambda^k({}^bT^*M)$ are *b*-**forms**. The standard differential extends to

$$d : {}^b\Omega^k(M) \to {}^b\Omega^{k+1}(M)$$

This defines a cohomology (the *b*-cohomology groups can be read from **Mazzeo-Melrose** ${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z)$).

Locally around any point $p \in Z$, a form $\omega \in {}^b\Omega^k(M)$ admits a decomposition as

$$\omega|_U = \frac{dz}{z} \wedge \alpha + \beta,$$

where $z : U \to \mathbb{R}$ is such that $z^{-1}(0) = Z \cap U$, and $\alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M)$.

b-symplectic manifolds

Definition

A b-**symplectic manifold** *is a triple* (M, Z, ω) *where* (M, Z) *is a* b*-manifold and* $\omega \in {}^b\Omega^2(M)$ *is* **closed** *and* **non-degenerate**.

Remark

Many results from symplectic geometry (for instance, Moser trick and topological restrictions) have their analogous in the *b*-symplectic context.

Modular vector field

Definition

Let (M, Z, ω) a b-symplectic manifold and let $\Omega \in \Omega^{2n}(M)$ a volume form.

The **modular vector field** is then defined as the derivation $v_{mod} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ given by $f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}$, where X_f is the Hamiltonian vector field of f.

Remark

The modular vector field is symplectic, so $\mathcal{L}_{v_{mod}}\omega = 0$.

Lemma

The singular hypersurface Z has a **cosymplectic structure**. In particular, it has a regular foliation $\mathcal{F}$ by symplectic leaves of maximal rank. Moreover, the modular vector field is tangent to Z and transverse to $\mathcal{F}$.

Normal vector field

Definition

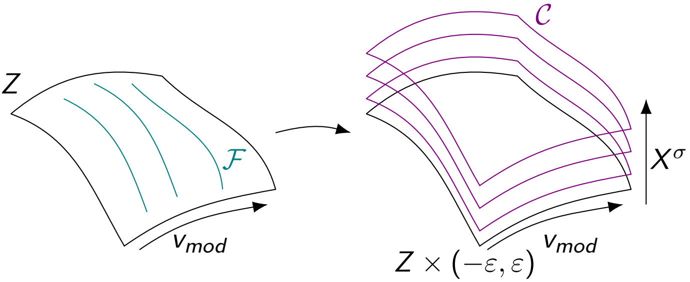
Consider (M, Z, ω) a *b*-symplectic manifold with Z orientable. Then there exists a **normal *b*-vector field** X^σ satisfying that

1. It is *symplectic*: $\mathcal{L}_{X^\sigma}\omega = 0$.
2. It is transversal to Z: $X_p^\sigma \notin TZ$ for all $p \in Z$.

Then, X^σ is conjugate with the *modular vector field*: $\omega(X^\sigma, v_{mod}) = 1$.
In local coordinates $(z, \theta, x_2, y_2, ..., x_n, y_n)$ the normal vector field has the expression $X^\sigma = z\frac{\partial}{\partial z}$.

Cosymplectic foliation

This choices induce a foliation of $(-\varepsilon, \varepsilon) \times Z$ by cosymplectic hypersurfaces:



Symplectic foliation

Focusing on the cosymplectic structure of Z, we have the following result regarding the symplectic foliation $\mathcal{F}$:

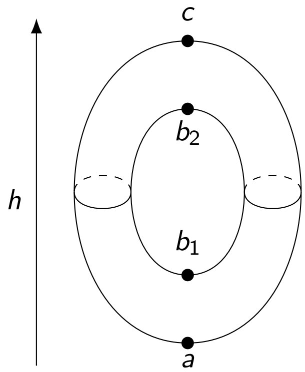
Theorem (Guillemin-Miranda-Pires)

If a leaf $L \in \mathcal{F}$ is **compact**, then all leaves of the foliation are isomorphic to L.
In addition, Z is then the total space of a fibration $f : Z \to S^1$ with fiber L.

Later, we will always assume that we are in this case.

Morse homology: Motivation (I)

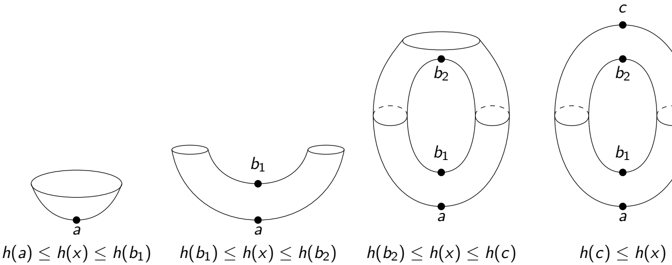
Torus embedded in $\mathbb{R}^3$:



Critical points in the torus

Morse homology: Motivation (II)

Region under the plane $h(x) = K$:



Formation of the torus

Morse homology: Summary

The Morse complex of a manifold M with respect to a Morse function $f : M \to \mathbb{R}$ is given by

- Generators (over $\mathbb{Z}_2$): Critical points of f.
- Classified by the (Morse) index.
- Connected by the flow of $-\mathrm{grad}_g f$ (g Riemannian metric).

The resulting homology $HM_\bullet(M)$ is isomorphic to the simplicial homology.

The Floer complex

We use the same idea as in the Morse complex

- Our domain is $\mathcal{L}M := \{x \in \mathcal{C}^\infty(S^1, M) \mid \text{contractible}\}$.
- Our function is the action functional $\mathcal{A}_H : \mathcal{L}M \longrightarrow \mathbb{R}$,

$$\mathcal{A}_H(x) := \int_0^1 H_t(x(t))dt - \int_{D^2} w^*\omega.$$

Remark

The critical points of $\mathcal{A}_H$ are precisely the 1-periodic orbits of X_H.

The periodic orbits are classified by the Conley-Zehnder index μ_{CZ}.

Almost complex structures

Definition

An **almost complex structure** J on M is a section of $TM \otimes T^*M$ such that

$$J^2 = -\mathrm{Id}.$$

It is **callibrated by** ω if

- $\omega(JX, JY) = \omega(X, Y)\ \forall X, Y \in \mathfrak{X}(M)$.
- $\omega(X, JX) > 0\ \forall X \in \mathfrak{X}(M)$.

This induces a Riemannian metric on M.

The Floer equation

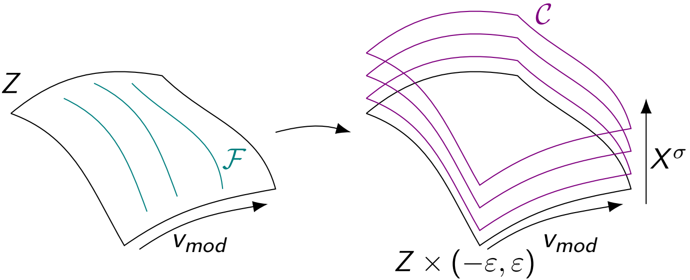
The (negative) gradient lines of $\mathcal{A}_H$ on (M, ω, J) are $u : \mathbb{R} \times \mathbb{S}^1 \to M$ such that satisfy the **Floer equation**:

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \mathrm{grad}_u H_t = 0$$

To connect critical points, we must impose the condition that

$$E(u) := \iint_{\mathbb{R} \times S^1} \left| \frac{\partial u}{\partial s} \right|^2 < +\infty$$

Where were we?


Z
$\mathcal{F}$
v_{mod}
$\mathcal{C}$
X^σ
$Z \times (-\varepsilon, \varepsilon)$
v_{mod}

Ah, yes, that...

Motivation

Our ambition is to construct **a Floer complex for *b*-symplectic manifolds**.

However, precautions must be taken when selecting admissible Hamiltonians on which to define such a complex.
In particular, we want to split the dynamics between Z and $M \backslash Z$.

We can imagine that we are defining a Floer complex for a non-compact manifold ($M \backslash Z$) with a *cosymplectic behavior at infinity*.

Admissible Hamiltonians (I)

Let us try to find relationships between admissible Hamiltonians $H : \mathbb{R} \times M \to \mathbb{R}$ and geometrical features of (M, Z, ω) (in a collar neighborhood of Z).

First, we want to prevent the flow of X_H to approach Z, so we want the "X^σ-component" of X_H to vanish.
Locally, this can be achieved by imposing that

$$\mathcal{L}_{v_{mod}} H = 0.$$

Admissible Hamiltonians (II)

Second, we want to have a contribution in the "v_{mod}-component" but somehow controlled. In particular, we want it to be constant.
We need to be careful about this, because...

Proposition

Let (W, Z, ω) a compact *b*-symplectic *surface* and $H_t : \mathbb{R} \times W \to \mathbb{R}$ such that $\mathcal{L}_{X^\sigma} H_t = k$ for some $k \in \mathbb{Z}$. Then, there exist 1-periodic orbits on each leaf $\sigma \in \mathcal{C}$ in the cosymplectic foliation.

Sketch of the proof...

- W.l.o.g. $Z = S^1$ and $(\mathcal{N}(Z), \omega) \cong \left((-\varepsilon, \varepsilon) \times S^1, \frac{dz}{z} \wedge d\theta\right)$. Then, $H_t(z, \theta) = k \log|z| + h_t(\theta)$ and

$$X_H(t, z, \theta) = k\frac{\partial}{\partial\theta} - z\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}.$$

- The flow of X_H is given by

$$\begin{cases} \theta(t) = \theta_0 + kt \\ z(t) = z_0 \exp\left(-\int_0^t \frac{\partial h_s}{\partial\theta}(\theta_0 + ks)ds\right) \end{cases},$$

which has a 1-periodic orbit if and only if $z(1) = z_0$, if and only if

$$F(\theta) := \int_0^1 \frac{\partial h_t}{\partial\theta}(\theta_0 + kt)dt = 0.$$

- Using Fubini's Theorem, integrating F over S^1,

$$\int_{S^1} F(\theta)d\theta = \int_{S^1}\int_0^1 \frac{\partial h_t}{\partial \theta}(\theta_0 + kt)dtd\theta =$$

$$\int_0^1\int_{S^1} \frac{\partial h_t}{\partial \theta}(\theta_0 + kt)d\theta dt = \int_0^1 \left[h_t(\theta + kt)\right]_{\theta=0}^{\theta=1} dt = 0$$

- Thus, there always exists some θ_0 such that X_H has a 1-periodic orbit when flowing from the point (z_0, θ_0).

Admissible Hamiltonians (III)

To prevent scenarios like this,

Third

Lemma

Let H_t given locally as $H_t(z, \theta, x) = k(t) \log |z| + h_t(x)$. Let $T \in \mathbb{R}$ denote the period of the modular vector field v_{mod} in the connected component of Z. Then, if

$$\int_0^1 k(t)dt \in (0, T)$$

the flow of X_H will have no 1-periodic orbits in a collar neighborhood of Z.

Summary of results

Definition

A Hamiltonian $H : S^1 \times M \to \mathbb{R}$ is **admissible** if

- It is invariant with respect to the modular vector field: $\mathcal{L}_{v_{mod}} H_t = 0$.
- It grows linearly in the normal direction: $\mathcal{L}_{X^\sigma} H_t = k(t)$ for some $k : \mathbb{R} \to \mathbb{R}$.
- Its Hamiltonian vector field exhibits no 1-periodic orbits in a tubular neighbourhood small enough around Z:

$$\int_0^1 k(t) dt \in (0, T).$$

$$H_t(z, \theta, x) = k(t) \log |z| + h_t(x)$$

To be continued...