The construction of the Floer complex

From Morse to Floer

Joaquim Brugués Mora

advisors

Eva Miranda (UPC) Sonja Hohloch (UAntwerp)

Universitat Politècnica de Catalunya

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Morse theory: motivation (I)

Torus embedded in $\mathbb{R}^3$:

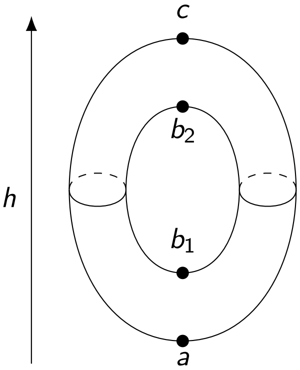


Figure: Critical points in the torus

Morse theory: motivation (II)

Region under the plane $h(x) = K$:

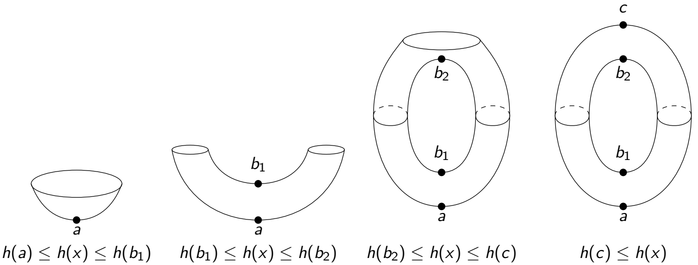


Figure: Formation of the torus

Morse Homology

Let M a smooth manifold and $f : M \to \mathbb{R}$ a Morse function.

- Generators (over $\mathbb{Z}_2$): Critical points of f.
- Classified by the index.
- Points are related by the flow of $-\mathrm{grad}_g f$ (g Riemannian metric).
- Boundary operator:

$$\partial_k p = \sum_{q \in \mathrm{Crit}_{k-1}(f)} n_g(p, q) q,$$

  with $n_g(p, q)$ the number of trajectories between p and q.
- Invariance: $HM_\bullet(M)$ does not depend on f or g.
- Morse homology is isomorphic to cellular homology.

Morse homology: Results

The homology $HM_\bullet(M)$ is a **topological invariant**.

Theorem (Morse inequalities)

Let $f : M \to \mathbb{R}$ a Morse function, c_k the number of critical points of index k, and β_k the k-th Betti number of M. Then,

$$c_k \geq \beta_k.$$

In particular,

$$\#\mathrm{Crit}(f) \geq \sum_{k=0}^{n} \beta_k.$$

Arnold conjecture

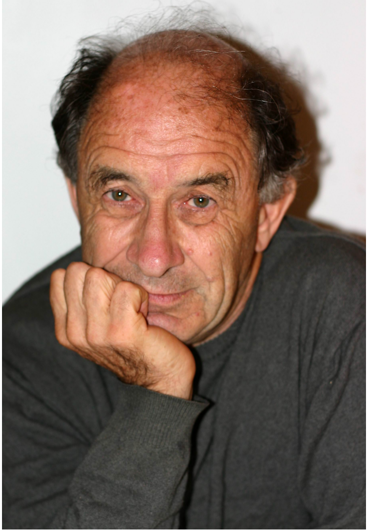


Figure: Vladimir Arnold

Symplectic manifolds

Definition

A **symplectic structure** on a $2n$-dimensional manifold M is a 2-form ω that is

- Closed: $d\omega = 0$.
- Non-degenerate $\omega(X, \cdot) \neq 0 \ \forall X \in \mathfrak{X}(M)$.

Motivation

It provides the setting to define Hamiltonian systems in manifolds. If $H : M \to \mathbb{R}$ is an energy function, then the vector field X_H with

$$\omega(X_H, Y) = -dH(Y) \ \forall Y \in \mathfrak{X}(M)$$

generalizes Hamilton's equations for H.

Arnold conjecture

Theorem (Arnold 1963)

Consider $H_t : \mathbb{R} \times M \to \mathbb{R}$ a time-dependent Hamiltonian. Then,

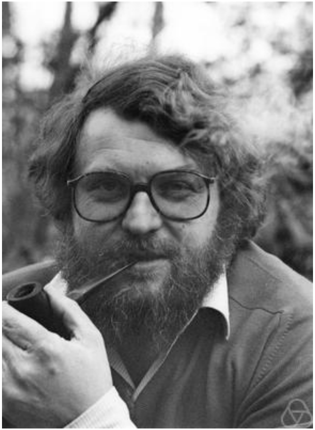
$$\#\{\text{1-periodic orbits of } X_{H_t}\} \geq \sum_{k=0}^{2n} \beta_k.$$

Remark

In the case of an autonomous Hamiltonian H, this is a consequence of the Morse inequalities.

Solution: Floer homology (1988)

(a) Andreas Floer



(b) Eduard Zehnder

Restrictions

We need to make some assumptions.

- **Non-degeneracy:** $\det(d_p\varphi^1_{X_{H_t}} - \mathrm{Id}) \neq 0$.
- **Asphericallity:** We assume that any sphere contained in M is contractible to a point (weaker assumptions can be made).
- **Contractible loops:** We only consider the loops $\mathcal{L}M = \{x \in \mathcal{C}^\infty(\mathbb{S}^1, M) \mid [x] = 1 \text{ in } \pi_1(M)\}$.

The Floer complex

- The complex is generated (over $\mathbb{Z}_2$) by the 1-periodic, non-degenerate and contractible solutions of

$$\dot{x} = X_{H_t}(x).$$

- We classify the periodic orbits using the **Conley-Zehnder index**

$$\mu_{CZ} : SP(M) \longrightarrow \mathbb{Z}.$$

The action functional

For all $x \in \mathcal{L}M$ we can take $w : \mathbb{D}^2 \to M$ such that $w|_{\partial \mathbb{D}^2} = x$. Asphericallity implies that $w \simeq w'$ for any w, w' with the same boundary x.

Definition

The **action functional** is $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$,

$$\mathcal{A}_H(x) = \int_0^1 H(x(t))dt - \int_{\mathbb{D}^2} w^*\omega$$

Proposition

A loop x is a critical point of $\mathcal{A}_H$ if and only if $\dot{x} = X_{H_t}(x)$.

The Floer equation (I)

Definition

An **almost complex structure** J on M is a section of $TM \otimes T^*M$ such that

$$J^2 = -\mathrm{Id}.$$

It is **callibrated by** ω if

- $\omega(JX, JY) = \omega(X, Y)\ \forall X, Y \in \mathfrak{X}(M)$.
- $\omega(X, JX) > 0\ \forall X \in \mathfrak{X}(M)$.

This induces a Riemannian metric on M.

The Floer equation (II)

The (negative) gradient lines of $\mathcal{A}_H$ on (M, ω, J) are $u : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to M$ such that satisfy the **Floer equation**:

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \mathrm{grad}_u H_t = 0$$

This is a generalization of **pseudoholomorphic curves** (studied by Gromov, 1985).
We need to impose that the energy is finite

$$E(u) = -\int_{\mathbb{R}} d\mathcal{A}_H \cdot \frac{\partial u}{\partial s} = \iint_{\mathbb{R}\times\mathbb{S}^1} \left|\frac{\partial u}{\partial s}\right|^2 < +\infty,$$

so that u connects critical points.

Boundary operator

Proposition

Let $\mathcal{M}(x, y)$ the finite energy solutions of the Floer equation connecting x and y.
Then, it is a compact manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y)$, and there is a free and proper $\mathbb{R}$-action on it, so we can quotient $\mathcal{M}(x, y) / \mathbb{R}$.

If $\mu_{CZ}(x) - \mu_{CZ}(y) = 1$, $\mathcal{M}(x, y) / \mathbb{R}$ is finite, so we define

$$n_J(x, y) = \# \left(\mathcal{M}(x, y) / \mathbb{R} \right),$$

and

$$\partial_k x = \sum_{\mu_{CZ}(y) = k-1} n_J(x, y) y.$$

Results

Theorem

The Floer homology $HF_\bullet(M)$ is a **topological invariant***: It does not depend on ω, H_t or J.*

Theorem

The Floer homology of a manifold is isomorphic to the Morse homology (with an index shifting):

$$HF_\bullet(M) \cong HM_{\bullet+n}(M).$$

This proves the Arnold conjecture for a wide class of manifolds.

Future work

- Is it possible to construct a Floer homology for b-symplectic manifolds?
- Which kind of bounds do we have on periodic orbits on b-symplectic manifolds?
- Can other Floer homologies be constructed in the b-geometry setting?

References

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