Towards a Floer homology for singular symplectic manifolds

Joaquim Brugués Mora[1]
Universitat Politècnica de Catalunya
joaquim.brugues@upc.edu

Abstract

Floer homology has grown to become a fundamental tool in a wide variety of fields in geometry and topology. Since Floer introduced his theory in the late eighties to prove the Arnold conjecture on fixed points of Hamiltonian symplectomorphisms, it has been used to introduce invariants in symplectic and contact geometry, to study the space of connections on 3-dimensional manifolds, to study knots on the 3-sphere and to disprove the existence of triangulations for topological manifolds of dimension ≥ 5, and new applications to geometrical and topological problems still appear steadily.

In our project, we intend to construct a homology akin to the Hamiltonian Floer homology in the context of b^m-symplectic geometry, a particular family of Poisson structures with controlled singularities in which we are often able to recover results from classical symplectic geometry. In this poster we present the setting of the problem, providing a definition of Hamiltonian Floer homology and the basic characteristics of b^m-symplectic geometry.

Model: Morse homology

Floer theory takes Morse homology as a model, trying to replicate its constructions in some infinite dimensional setting. Morse homology studies the critical points of a Morse function defined on a smooth manifold, $f : M \to \mathbb{R}$, by building a homology on them. The building blocks of the homology are the following:

- The generators of the groups (over either $\mathbb{Z}$ or $\mathbb{Z}_2$) are the critical points of f. As f is a Morse function, these critical points are isolated and, if M is compact, there is a finite number of them.
- The critical points are classified by an index. In this case, the index of p is the number of negative eigenvalues of the Hessian of f.
- The boundary operator on the complex, $\partial_k : CM_k(M,f) \to CM_{k-1}(M,f)$ is defined by finding a way to connect critical points. This is acomplished by the flow equation,

$$\dot{\gamma}(t) = -\text{grad} f(\gamma(t)).$$

A generic choice of a Riemannian metric will allow the flow of the vector field $-\text{grad}f$ to define a proper boundary map $\partial_\bullet$.

The resulting homology, $HM_\bullet(M)$, does not depend on f or g (the Morse function or the Riemannian metric), and is therefore a topological invariant. Moreover, it is isomorphic to the singular homology. This yields the **Morse inequalities**:

$$\#\{\text{critical points of index } k \text{ of } f\} \geq \beta_k,$$

where β_k is the k-th Betti number of M.

Hamiltonian Floer homology

Let (M,ω) a compact symplectic manifold, and take $H_t : \mathbb{R}\times M \to \mathbb{R}$ a time-dependent Hamiltonian. This function induces a (time dependent) vector field X_{H_t} in M, and this in turn induces a flow $\varphi^t_{H_t} : \mathbb{R}\times M \to M$ by Hamiltonian symplectomorphisms. Arnold proposed the following hypothesis in 1963:

The number of fixed points of $\varphi^1_{H_t}$ greater or equal to the sum of the Betti numbers of M.

This can be seen as a lower bound on the number of periodic orbits af a prescrived period $T > 0$ (we take $T = 1$ without losing generality). When H is $\mathcal{C}^2$-small enough or not dependent on time this is already true due to the Morse inequalities (as fixed points are particular, trivial cases of periodic orbits).

During the following years the main project was to build an *infinite-dimensional Morse homology* using the manifold $\mathcal{C}^\infty(\mathbb{S}^1, M)$ and the **action functional** as Morse funcion,

$$\mathcal{A}_H(x) = \int_{\mathbb{S}^1} (H_t(x(t)) - \lambda(\dot{x}(t)))dt.$$

One can define the action functional either assuming that ω has a primitive λ (such that $d\lambda = \omega$), which is never the case for compact symplectic manifolds, or by imposing certain topological conditions on M. For instance, one may impose *symplectic asphericallity*,

$$\int_{\mathbb{S}^2} \psi^*\omega = 0 \,\forall \psi : \mathbb{S}^2 \to M.$$

The critical points of $\mathcal{A}_H$ are precisely the 1-periodic orbits of X_{H_t} that we are interested in, so a Morse-like theory seems a good strategy. However, these critical points may have infinite index and co-index, so a direct application of Morse homology is not possible. Moreover, when we introduce some metric (this means, topology) in $\mathcal{C}^\infty(\mathbb{S}^1, M)$ in order to get a flow and connect critical points we may find that the trajectories of the flow do not satisfy the compactness conditions that are crucial to define the boundary operator in Morse homology.

Floer found a way to overcome the second issue, choosing to work with the L^2 topology instead of a finer one in $\mathcal{C}^\infty(\mathbb{S}^1, M)$. In order to define it, we introduce a **compatible almost complex structure**, this means, a tensor field

$$J \in \Gamma(T^*M \otimes TM) \text{ such that } J_x \circ J_x = -\text{Id}\,\forall x \in M,$$

and $\omega_x(u, J_x v)$ defines a Riemannian metric for all $x \in M$.

This Riemannian metric can be used on M to define a metric on $\mathcal{C}^\infty(\mathbb{S}^1, M)$, and from there the gradient of $\mathcal{A}_H$ can be computed. If one studies the flow of the (infinite dimensional) vector field $-\text{grad}\mathcal{A}_H$, the **Floer equation** can be deduced,

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \text{grad} H_t(u) = 0,$$

where u is a smooth function $u : \mathbb{R}\times\mathbb{S}^1 \to M$. This equation is very similar to the Cauchy-Riemann equations in the context of almost complex manifolds, but with an extra term given by the gradient of the Hamiltonian.

Solutions of the Floer equation by themselves are not enough to define the boundary operator, but we also need to introduce the energy on the set of solutions, given by

$$E(u) = -\int_{\mathbb{R}} u^* d\mathcal{A}_H = \int_{\mathbb{R}\times\mathbb{S}^1} \left|\frac{\partial u}{\partial s}\right|^2 dt ds.$$

If we impose that $E(u) < +\infty$, then it is possible to prove that the solutions of the Floer equation do connect the critical points of the action functional, this means, the 1-periodic orbits of X_H.

Another issue that needs to be addressed is the grading of the homology groups. This is solved by the introduction of the **Conley-Zehnder index** μ_{CZ}, which intuitively measures how much does a path of symplectic linear transformations "twist" through a loop (in our case, the linear transformations are $D\varphi^t_{X_H}$, the flow of the Hamiltonian vector field. With all of these elements it is possible to define the groups of the Floer complex,

$$CF_k(M,H) = \langle p \in M \mid p = \varphi^1_{X_H}(p) \text{ and } \mu(D\varphi^t_{X_H}(p)) = k\rangle_{\mathbb{Z}_2},$$

and a boundary map

$$\partial^J_k : CF_k(M,H) \longrightarrow CF_{k-1}(M,H)\,,$$

defining a complex and, therefore, inducing a homology,

$$HF_k(M,H,J) = \frac{\text{Ker}\partial_k}{\text{Im}\partial_{k+1}}.$$

One can then go on to prove that the homology does not actually depend on the Hamiltonian function H_t used to define the complex nor on the almost complex structure J used to define the metric. Moreover, as we mentioned earlier, the homology coincides with the Morse homology (with a shift in the index) for certain choices of H, and therefore

$$HF_\bullet(M) \cong HM_{\bullet+n}(M).$$

b-symplectic manifolds

Let us consider a codimension one submanifold $Z \subset M$, and let ${}^b\mathfrak{X}(M)$ denote the space of vector fields tangent to Z. In the context of the b-calculus it is possible to explore the dynamics of such vector fields sistematically. Using the Serre-Swan theorem, we can define the vector bundle ${}^bTM$ so that ${}^b\mathfrak{X}(M)$ are its sections. It is called the **b-tangent bundle**

In this context one can define b-differential forms and the differential operator, extending the usual differential forms but allowing also for a singularity to happen on Z. Thus, it is possible to define a b-2-form, closed and non-degenerate, that defines a b-symplectic structure. The model for such a b-symplectic form is

$$\omega = \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i,$$

where $Z = \{x_1 = 0\}$ locally. We have Darboux theorem in the context of b-symplectic forms, so such a local model always exists.

b^m-symplectic manifolds

A further generalization of b-symplectic structures are the ones in which we allow a singularity of order m to happen at Z or, equivalently, we only take into account the vector fields that are tangent to Z and that vanish at order m at some direction transverse to Z. The Darboux coordinates in this case yield the local form

$$\omega = \frac{dx_1}{x_1^m} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

Future work

The objective of our project is to construct the Floer homology as in the case of the Hamiltonian vector fields in the context of b^m-symplectic manifolds. There are several strategies that we may employ to accomplish this.

A first one may be to approximate the homology groups by a limit, in some sense (by taking a limit of symplectic forms $\omega_\varepsilon \to \omega$ to a b^{2k}-symplectic form), as in [4].

A different possibility would consist on studying the behaviour of J-holomorphic curves for some almost complex structure adapted to ω and defined on ${}^bTM$.

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