A Hamiltonian Floer complex for b-manifolds

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Abstract

Since the beginnings of the development of Floer theory there has been an interest to adapt its homology to non-compact manifolds. In our project we aim at building a Floer complex for the open part of b-symplectic manifolds, using a cosymplectic foliation in a tubular neighbourhood of of the singular hypersurface.

1 Hamiltonian Floer homology

Theorem 1 (Arnold Conjecture). Let (M, ω) a compact symplectic manifold and H a non-degenerate Hamiltonian. Then,

$$\#\{\text{1-periodic orbits of } X_H\} \geq \sum_{k=0}^{2n} \beta_k.$$

To prove this conjecture, Conley, Zehnder and Floer employed a construction analogous to the Morse complex in infinite variables.

Definition 1. Let $\mathcal{L}M = \{\gamma \in \mathcal{C}^\infty(S^1, M) \mid \gamma \text{ is contractible.}\}$. The *action functional* is the map $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ given by

$$\mathcal{A}_H(x) = \int_{S^1} H_t(x(t))dt - \int_B u^*\omega,$$

where $u : D^2 \to M$ is such that $u(e^{it}) = x(t)$.

Our complex can then be defined in an analogous way to Morse theory, using a generic choice of almost complex structure J.

- The generators of $CF_\bullet(M, \omega, H, J)$ are the critical points of $\mathcal{A}_H$. This is finite because of the conditions we can impose on H.
- The groups are classified by a *Conley-Zehnder index* μ_{CZ}. For the particular case of fixed points of X_H, this corresponds to a shift of the Morse index of H.
- The boundary operator $\partial_k : CF_k(M, \omega, H, J) \to CF_{k-1}(M, \omega, H, J)$ is defined using a connection between 1-periodic orbits. This is given by the *Floer equation*,

$$\frac{\partial u}{\partial s} + J_u\left(\frac{\partial u}{\partial t} - X_H\right) = 0$$

where $u : \mathbb{R} \times S^1 \to M$, with the *energy condition*,

$$\int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 dtds < +\infty$$

Theorem 2. The induced *Floer homology* $HF_\bullet(M)$ is independent of ω, H and J. Moreover, $HF_\bullet(M) \cong HM_{\bullet+n}(M)$.

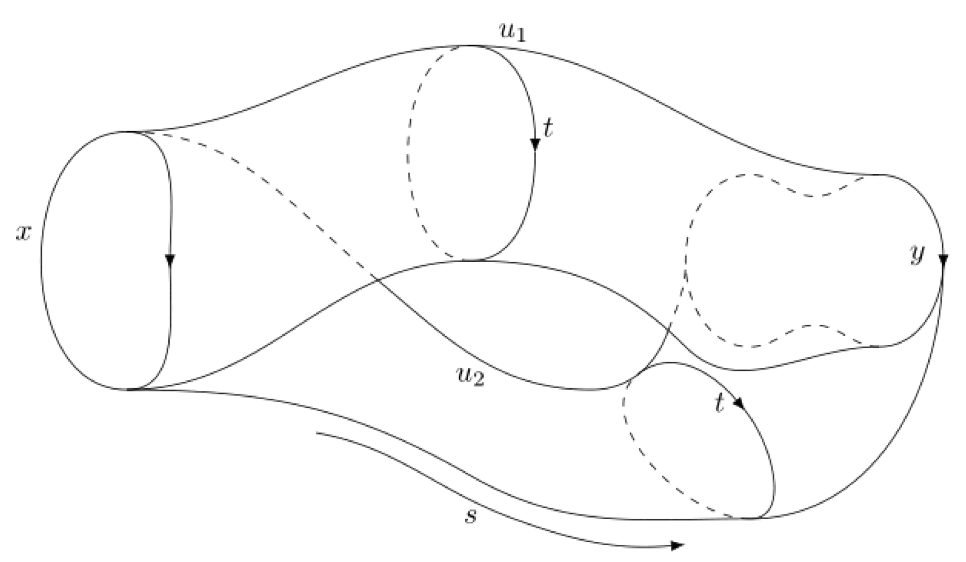

Figure 1: Sketch of solutions to the Floer equation

2 b-symplectic manifolds

Definition 2. Let M a smooth manifold and $Z \subset M$ an embedded codimension 1 submanifold. We denote by ${}^b\mathfrak{X}(M)$ the set of vector fields tangent to Z. It induces a vector bundle, the *b-tangent bundle*, denoted as ${}^bTM \to M$.

In this context we can define forms and the differential, so we can define the complex of b-forms, ${}^b\Omega^\bullet(M)$.

Definition 3. A *b-symplectic manifold* is a b-pair (M^{2n}, Z^{2n-1}) endowed with a 2-b-form $\omega \in {}^b\Omega^2(M)$ that is closed and non-degenerate.

If $p \in Z$ and U is a small neighbourhood in M, then we can choose coordinates such that

$$\omega|_U = \frac{dz}{z} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

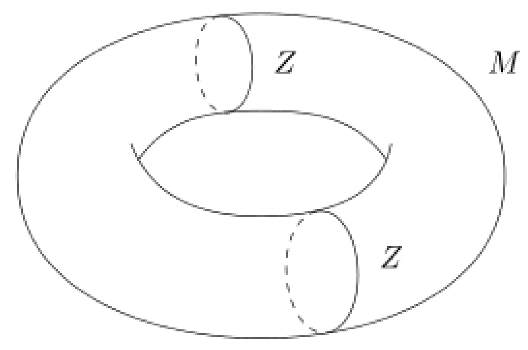

Figure 2: A sketch of a b-manifold

Z has an structure of a *cosymplectic manifold*, so there exist $\alpha \in \Omega^1(Z)$, $\beta \in \Omega^2(Z)$ closed such that $\alpha \wedge \beta^{n-1} \neq 0$.

Definition 4.

- The *normal vector field* to Z is the symplectic vector field $X^\sigma \in {}^b\mathfrak{X}(M)$ such that it is transverse to Z at every point.
- The *modular vector field* is an extension to M of the vector field transverse to the symplectic foliation $\mathcal{F}$ in Z. We denote it like v_{mod}, and it can be chosen so that $v_{mod} \in {}^b\mathfrak{X}(M)$ and it is symplectic.

Using both vector fields we can define a foliation of a tubular neighbourhood of Z by *cosymplectic hypersurfaces*.

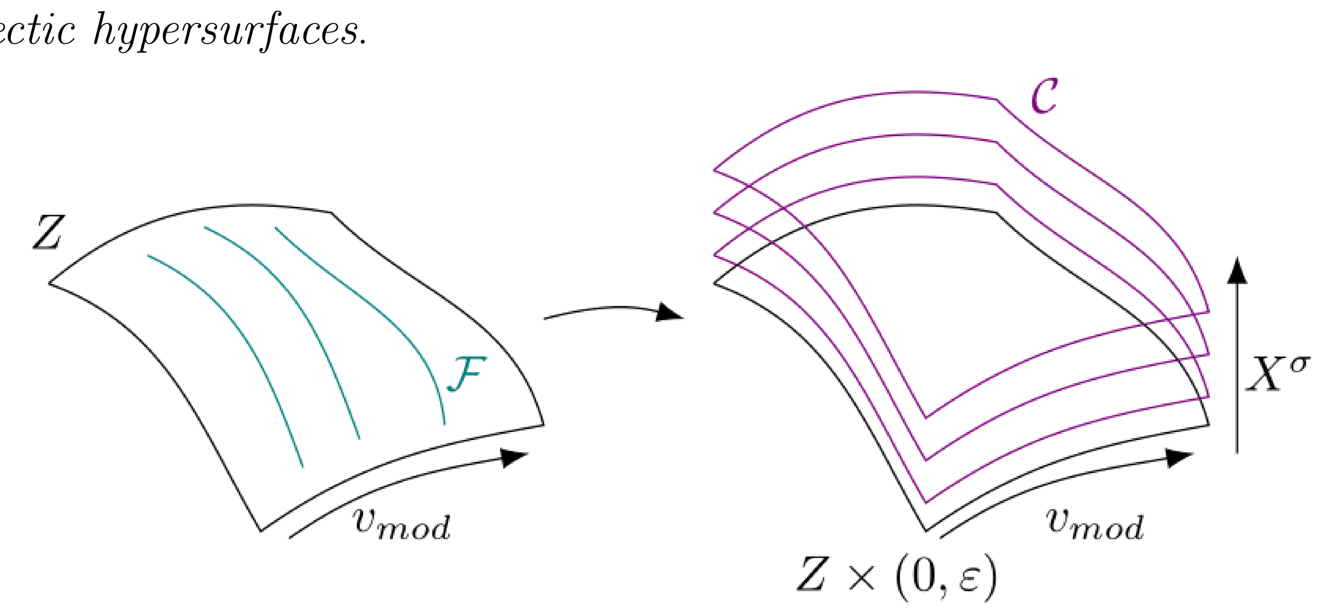

Figure 3: A foliation by cosymplectic hypersurfaces

3 A singular Floer homology

Definition 5. An *admissible Hamiltonian* $H : S^1 \times M \to \mathbb{R}$ is a non-degenerate Hamiltonian such that $\mathcal{L}_{v_{mod}} H_t = 0$, $\mathcal{L}_{X^\sigma} H_t = k(t)$, and X_H has no 1-periodic orbits in a tubular neighbourhood of Z.

Locally, they have the form

$$H_t(z, \theta, x) = k(t)\log|z| + h_t(x).$$

Lemma 3. If $\int_{S^1} k(t)dt \notin T\mathbb{Z}$, where T is the modular weight of Z, then X_H has no 1-periodic orbits in a neighbourhood of Z.

Definition 6. The groups of the b-Floer complex are given by

$$CF_k(M, Z, \omega, H) = \langle \text{1-periodic contractible orbits of } X_H \text{ with } \mu_{CZ}(\gamma) = k\rangle_{\mathbb{Z}_2}$$

Conjecture 1. Let (M, Z, ω) a b-symplectic manifold and H a non-degenerate Hamiltonian with suitable restrictions. Then,

$$\#\{\text{1-periodic orbits of } X_H\} \geq \sum_{k=0}^{2n} \dim H_k(M, Z),$$

the relative homology of M with respect to Z.

Proposition 4. Let u be a solution of the Floer equation on $\mathcal{N} \cong Z \times (0, \varepsilon)$. Let $f : \mathcal{N} \to \mathbb{R}$ given by $f(z, x) = -\log|z|$. Then, $\Delta(f \circ u) = 0$.

Corollary 5. For every two 1-periodic orbits x and y, the moduli space $\mathcal{M}(x, y)$ of solutions of the Floer equation connecting x to y is a compact manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y) - 1$.

This allows us to define the boundary map $\partial_k : CF_k(M, Z, \omega, H, J) \to CF_{k-1}(M, Z, \omega, H, J)$ as in the usual Floer complex.

Definition 7. The *b-Floer homology* is the homology of this complex,

$$HF_\bullet(M, Z) = \frac{\ker(\partial_\bullet)}{\operatorname{im}(\partial_{\bullet+1})}.$$

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