Constructing a singular Floer homology

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This is a joint work with Eva Miranda and Cédric Oms.

Abstract

Floer homology has grown to become a fundamental tool in a wide variety of fields in geometry and topology. Since Floer introduced his theory in the late eighties to prove the Arnold conjecture on fixed points of Hamiltonian symplectomorphisms, there has been a steady output of results using this tool, and new applications to geometrical and topological problems still appear.

In our project, we intend to construct a homology akin to the Hamiltonian Floer homology in the context of b-symplectic geometry, a particular family of Poisson structures with controlled singularities in which we are often able to recover results from classical symplectic geometry.

1 Model: Morse Theory

Definition 1. A *Morse function* $f : M \to \mathbb{R}$ defined on a smooth manifold is such that all of its critical points are non-degenerate.

For such a function f and a generic Riemannian metric g we can construct a complex $CM_\bullet(M, f, g)$ over either $\mathbb{Z}$ or $\mathbb{Z}_2$.

- The generators of the complex are $\text{Crit}(f)$. Since f is a Morse function, its critical points are isolated and, if M is compact, there is a finite number of them.
- The critical points are classified by an index. In this case, the index of p is the number of negative eigenvalues of the Hessian of f.
- The boundary operator $\partial_k : CM_k(M, f, g) \to CM_{k-1}(M, f, g)$ is defined by finding a way to connect critical points. This is accomplished by the flow equation,
$$\dot{\gamma}(t) = -\text{grad}_{\gamma(t)} f.$$
A generic choice of a Riemannian metric will allow the flow of the vector field $-\text{grad} f$ to define a proper boundary map $\partial_\bullet$.

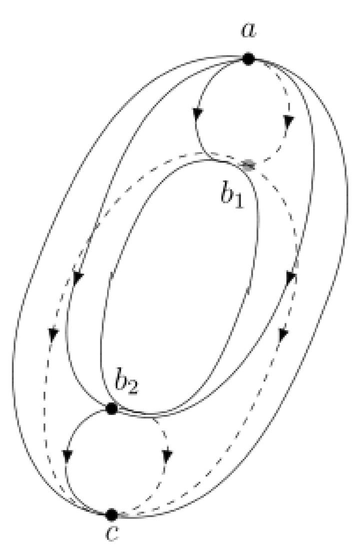

Figure 1: Construction of the Morse complex for the torus

Theorem 1. The induced *Morse homology* $HM_\bullet(M)$ is independent of f and g, and therefore it is a topological invariant. Moreover, $HM_\bullet(M) \cong H_\bullet(M)$, the singular homology.

Corollary 1. *Morse inequalities*:
$$\#\{\text{critical points of index } k \text{ of } f\} \geq \beta_k,$$
where β_k is the k-th Betti number of M.

2 Hamiltonian Floer homology

Definition 2. A *symplectic manifold* (M^{2n}, ω) is a smooth manifold endowed with a closed and non-degenerate form $\omega \in \Omega^2(M)$.

In this context, for any function $H : S^1 \times M \to \mathbb{R}$ we can define its *symplectic gradient* as the only vector field $X_H \in \mathfrak{X}(M)$ such that
$$i_{X_H}\omega = -dH.$$
In this context, there is a conjecture proposed by V.I. Arnold on 1963 (now proved) ana–logous to Morse inequalities,

Theorem 2 (Arnold Conjecture). Let (M, ω) a compact symplectic manifold and H a non-degenerate Hamiltonian (which is an open condition). Then,
$$\#\{\text{1-periodic orbits of } X_H\} \geq \sum_{k=0}^{2n} \beta_k.$$
To prove this conjecture, Conley, Zehnder and Floer employed a construction analogous to the Morse complex in infinite variables.

Definition 3. Let $\mathcal{L}M = \{\gamma \in \mathcal{C}^\infty(S^1, M) \mid \gamma \text{ is contractible.}\}$. The *action functional* is the map $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ given by
$$\mathcal{A}_H(x) = \int_{S^1} H_t(x(t))dt - \int_B u^*\omega,$$
where $u : D^2 \to M$ is such that $u(e^{it}) = x(t)$.

Our complex can then be defined in an analogous way to Morse theory, using a generic choice of almost complex structure J.

- The generators of $CF_\bullet(M, \omega, H, J)$ are the critical points of $\mathcal{A}_H$. This is finite because of the conditions we can impose on H.
- The groups are classified by a *Conley-Zehnder index* μ_{CZ}. For the particular case of fixed points of X_H, this corresponds to a shift of the Morse index of H.
- The boundary operator $\partial_k : CF_k(M, \omega, H, J) \to CF_{k-1}(M, \omega, H, J)$ is defined using a connection between 1-periodic orbits. This is given by the *Floer equation*,
$$\begin{cases} \frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \text{grad}_u H_t = 0 \\ \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 dtds < +\infty \end{cases}$$
where $u : \mathbb{R} \times S^1 \to M$.

Theorem 3. The induced *Floer homology* $HF_\bullet(M)$ is independent of ω, H and J. Moreover, $HF_\bullet(M) \cong HM_{\bullet+n}(M)$.

3 b-symplectic manifolds

Definition 4. Let M a smooth manifold and $Z \subset M$ an embedded codimension 1 submanifold. We denote by ${}^b\mathfrak{X}(M)$ the set of vector fields tangent to Z. It induces a vector bundle, the *b-tangent bundle*, denoted as ${}^bTM \to M$.

In this context we can define forms and the differential, so we can define the complex of b-forms, ${}^b\Omega^\bullet(M)$.

Definition 5. A *b-symplectic manifold* is a b-pair (M^{2n}, Z^{2n-1}) endowed with a 2-b-form $\omega \in {}^b\Omega^2(M)$ that is closed and non-degenerate.

If $p \in Z$ and U is a small neighbourhood in M, then we can choose coordinates such that
$$\omega|_U = \frac{dz}{z} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$
As with symplectic manifolds, we can define symplectic gradients for functions $H : S^1 \times M \to \mathbb{R}$. Then, $X_H \in {}^b\mathfrak{X}(M)$.

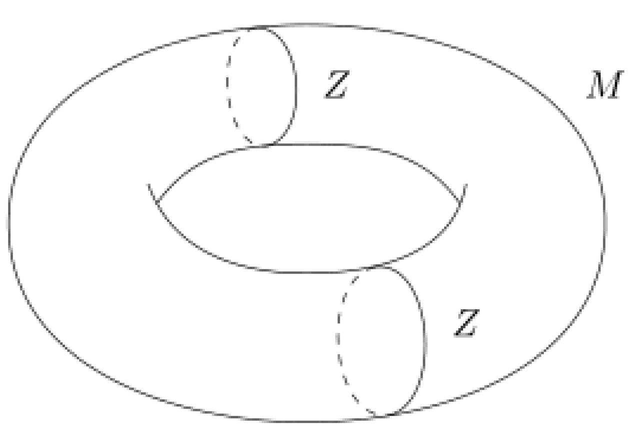

Figure 2: A sketch of a b-manifold

4 A singular Floer homology

Conjecture 1. Let (M, Z, ω) a b-symplectic manifold and H a non-degenerate Hamiltonian with suitable restrictions. Then,
$$\#\{\text{1-periodic orbits of } X_H\} \geq \sum_{k=0}^{2n} \dim H_k(M, Z),$$
the relative homology of M with respect to Z.

The restrictions on H are designed in this case to ensure that there are no periodic orbits in a small enough semilocal neighbourhood of Z, and that the solutions of the Floer equation also stay away from this neighbourhood.

Definition 6. The *normal vector field* to Z is the symplectic vector field $X^\sigma \in {}^b\mathfrak{X}(M)$. A Hamiltonian is then *admissible* if there exists a non-zero constant k such that
$$\mathcal{L}_{X^\sigma} H = k$$
and small enough so that X_H has no periodic orbits.

References:

[1] M. Audin, M. Damian. *Morse theory and Floer homology.* London: Springer, 2014.

[2] U. Frauenfelder, F. Schlenk. *Hamiltonian dynamics on convex symplectic manifolds.* Israel Journal of Mathematics 159.1 (2007): 1-56.

[3] V. Guillemin, E. Miranda, A.R. Pires, *Symplectic and Poisson geometry on b-manifolds, Adv. Math.* 264, pp. 864-896 (2014).

[4] R. Melrose, *The Atiyah Patodi Singer Index Theorem*, Res. Not. Math., A. K. Peters.