Exercise Sheet 1
Multivariate Calculus

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The exercises must be handed in on Friday, February 26th at the latest.

1 Matrix norms (8 points)

We know that on matrix spaces $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ we can define the operator norms as $\|A\|_{\text{op}} = \sup\{\|A(x)\| \mid \|x\| \leq 1\}$, where $\|\cdot\|$ is a norm in $\mathbb{R}^m$ and $\mathbb{R}^n$ (for this exercise we consider the same norm for both spaces).

Recall the definitions of the sum, Euclidean and supremum norms in $\mathbb{R}^n$:

$$\|x\|_\Sigma = \sum_{i=1}^n |x_i|, \ \|x\|_E = \sqrt{\sum_{i=1}^n x_i^2}, \ \|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

Show that

1. $\|A\|_\Sigma = \max\limits_{1\leq j\leq m} \left(\sum\limits_{i=1}^n |a_{i,j}| \right)$.
2. $\|A\|_\infty = \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^m |a_{i,j}| \right)$.
3. $\|A\|_E = \sqrt{\sigma_M(A)}$, where $\sigma_M(A)$ denotes the largest eigenvalue of the matrix $A^T A$.
4. Compute each of these norms for the matrix $A = \begin{pmatrix} 1 & \frac{5}{2} & \frac{9}{2} \\ 0 & 4 & 5 \\ 0 & -\frac{5}{2} & -\frac{7}{2} \end{pmatrix}$.

(Hint: A possible way to prove the identities is to show that the norm is lower or equal to the target value, and then prove that there is a vector such that the equality is reached)

2 Induced norms (4 points)

We know that a norm always induces a metric in a vector space, but that the converse is not necessarily true. What follows is a way to deduce under which conditions is a metric induced by a norm.

Consider V a vector space. Show that a metric d is induced by a norm if and only if these two properties are satisfied:

1. **Translation invariance:** For any $x, y, z \in V$ we have $d(x + z, y + z) = d(x, y)$.
2. **Homogeneity:** For any $\alpha \in \mathbb{R}$ and $x, y \in V$ we have $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

3 Topology of set addition (4 points)

Consider A, B two subsets of $\mathbb{R}^n$. Let $A + B := \{x + y \mid x \in A, y \in B\}$, and suppose that A is open.

1. Prove that, given $y \in \mathbb{R}^n$, the set $A + \{y\}$ is open.
2. Prove that $A + B$ is open.

4 Topology of set product (4 points)

Consider A, B two subsets of $\mathbb{R}$. Let $A \cdot B := \{xy \mid x \in A, y \in B\}$.

1. Prove that if A is open and $B \subset (0, +\infty)$, then $A \cdot B$ is open.
2. Show that if A is open $A \cdot B$ is not always open.