Geodetic and hull numbers of strong products of graphs *

J. Cáceres[1], C. Hernando[2], M. Mora[2], I.M. Pelayo[2], and M.L. Puertas[1]

[1] Universidad de Almería, {jcaceres,mpuertas}@ual.es
[2] Universitat Politècnica de Catalunya,
{carmen.hernando,merce.mora,ignacio.m.pelayo}@upc.edu

Abstract. A set S of vertices of a connected graph G is *convex*, if for any pair of vertices $u, v \in S$, every shortest $u-v$ path is contained in S. The *convex hull* $[S]$ of a set of vertices S is defined as the smallest convex set in G containing S. The set S is *geodetic*, if every vertex of G lies on some shortest path joining two vertices in S, and it is said to be a *hull set* if its convex hull is $V(G)$. The *geodetic* and the *hull numbers* of G are the minimum among all cardinalities of geodetic and hull sets, respectively. In this work, we determine some bounds and exact values of the geodetic and the hull numbers of strong products of graphs.

Key words: graph, geodetic set, convex set, hull set, strong product

1 Introduction

Classic convexity can be extended to graphs in a natural way by considering shortest paths, also called geodesics: a set S of vertices of a graph is convex if it contains all the vertices lying in some geodesic with endpoints in S and the convex hull of a set S of vertices is the minimum convex set containing S. Farber and Jamison [11] characterized the graphs such that every convex set is the convex hull of its simplicial vertices (i.e. vertices such that its neighborhood induces a complete graph). This rebuilding problem can be studied for general graphs from different points of view [15]. Geodetic and hull numbers give how many vertices are needed, at least, to rebuild the vertex set of a graph by using the closed interval and the convex hull operations respectively.

Both of these sides of the rebuilding problem have been studied for different graph classes obtained by means of graph operations. For example in cartesian products [1,5,14], compositions [6] and joins [7] of graphs. In this work, we develop these topics for strong products of graphs.

* Research partially supported by projects JA-FQM-305, P06-FQM-01649, 2009SGR1040, 2009SGR 1387, MTM2008-06620-C03-01, MTM2008-05866- C03-01, MTM2009-07242.

2 Preliminaries

In this work we consider only finite, simple and connected graphs. We denote respectively by $V(G)$ and $E(G)$ the set of vertices and edges of the graph G. We write $uv \in E(G)$ if u, v are adjacent vertices of G. Given two vertices u, v in a graph G let $d_G(u, v)$ denote the distance between u and v in G and $\text{diam}(G)$ denote the diameter of the graph G. When there is no confusion, subscripts will be omitted. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [16]. An $x-y$ path of length $d(x, y)$ is called an *$x-y$ geodesic*. The *closed interval* $I_G[x, y]$ consists of all vertices lying in some $x-y$ geodesic of G. For $S \subseteq V(G)$, the *geodetic closure* $I_G[S]$ of S is the union of all closed intervals $I_G[u, v]$ over all pairs $u, v \in S$, i.e. $I_G[S] = \bigcup_{u,v\in S} I_G[u, v]$. We omit subscripts if there is no confusion. A set S of vertices of G is *geodetic* if $I[S] = V(G)$ and *convex* if $I[S] = S$. The *convex hull* of $S \subseteq V(G)$ is the smallest convex set containing S and is denoted by $CH(S)$ [10]. If we define $I^0[S] = S$ and $I^r[S] = I[I^{r-1}[S]]$ for every $r \geq 1$, then convex hull of S can be obtained as $CH(S) = \cup_{r\geq 0} I^r[S]$. A set $S \subseteq V(G)$ is said to be a *hull set* if its convex hull is the whole vertex set $V(G)$, i.e. $CH(S) = V(G)$. The *geodetic number* and the *hull number* of a graph G are respectively the minimum cardinality among all geodetic sets and hull sets [10,12]. We denote them by $g(G)$ and $h(G)$. Certainly, every geodetic set is a hull set, and hence, $h(G) \leq g(G)$. It is easy to determine the geodetic and hull numbers of certain families of graphs (see Table 1).

G	P_n	C_{2l}	C_{2l+1}	T_n	K_n	$K_{p,q}$	$S_{1,p}$	$W_{1,p}$
$h(G)$	2	2	3	#leaves	n	2	p	$\lceil p/2 \rceil$
$g(G)$	2	2	3	#leaves	n	$\min\{4, p, q\}$	p	$\lceil p/2 \rceil$

Table 1. Hull and geodetic number of paths, cycles, trees, cliques, bicliques, stars and wheels.

A vertex v of a graph G is a *simplicial vertex* if the subgraph induced by its neighborhood $N(v) = \{u : uv \in E(G)\}$ is a clique. The set of all simplicial vertices of a graph G is denoted by $Ext(G)$. It is easily seen that every hull set, and hence every geodesic set, must contain the set $Ext(G)$.

The *strong product* $G \boxtimes H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H) = \{(g, h) : g \in V(G), h \in V(H)\}$ in which vertices (g, h) and (g', h') are adjacent whenever (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$, or (3) $gg' \in E(G)$ and $hh' \in E(H)$.

The distance between two vertices of the strong product can be determined as follows.

Lemma 1. *[13] Let G and H be graphs and $(g, h), (g', h') \in V(G \boxtimes H)$. Then,*

$$d_{G\boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\},$$

and consequently

$$\mathrm{diam}(G\boxtimes H) = \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}.$$

The projection onto G of a subset $S \subseteq V(G\boxtimes H)$ is the set $\mathrm{Pr}_G(S)$ of vertices $g \in V(G)$ for which there exists a vertex $(g,h) \in S$. Similarly we define the projection $\mathrm{Pr}_H(S)$ of S onto H. The projection onto G (resp. onto H) of a $(g,h)-(g',h')$ path gives a sequence of vertices that is not necessarily a path. If we delete possibly consecutive repeated vertices we obtain a walk in G.

Lemma 2. *Let G and H be graphs and $(g,h),(g',h') \in V(G\boxtimes H)$ such that $d_{G\boxtimes H}((g,h),(g',h')) = d(g,g')$, then the projection of the vertices of a $(g,h)-(g',h')$ geodesic in $G\boxtimes H$ onto G is a $g-g'$ geodesic in G.*

Proof. Consider the projections onto G of a $(g,h)-(g',h')$ geodesic in $G\boxtimes H$. We obtain a sequence of $d(g,g')$ vertices such that two consecutive vertices are equal or adjacent in G. This sequence is a $g-g'$ geodesic since it has no repeated vertices, otherwise it contains a $g-g'$ path of length less than $d(g,g')$, which is a contradiction. □

3 Geodetic and hull sets of the strong product

In this section, we study the behavior of the geodetic number with respect to the strong product operation for graphs in terms of its factors. More precisely, we obtain bounds, and we give some examples showing that some of them are sharp.

Firstly, we relate closed intervals in the strong product of two graphs to closed intervals in factor graphs, which is a key result to study both geodetic and hull numbers.

Lemma 3. *Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ for graphs G and H, then*

$$I_G[S_1] \times I_H[S_2] \subseteq I_{G\boxtimes H}[S_1 \times S_2].$$

Proof. If $(x,y) \in I_G[S_1] \times I_H[S_2]$, there exist vertices $g,g' \in S_1$ and $h,h' \in S_2$ such that $d_G(g,x) + d_G(x,g') = d_G(g,g')$ and $d_H(h,y) + d_H(y,h') = d_H(h,h')$. We may assume without loss of generality that $d_G(g,x) \leq d_G(x,g')$, $d_H(h,y) \leq d_H(y,h')$ and $d_G(g,x) \leq d_H(h,y)$. Hence

$$\begin{aligned} d_{G\boxtimes H}((g,h),(g,h')) &= d_H(h,h') = d_H(h,y) + d_H(y,h') \\ &= \max\{d_G(g,x), d_H(h,y)\} + \max\{d_G(x,g), d_H(y,h')\} \\ &= d_{G\boxtimes H}((g,h),(x,y)) + d_{G\boxtimes H}((x,y),(g,h')), \end{aligned}$$

implying that $(x,y) \in I_{G\boxtimes H}[(g,h),(g,h')] \subseteq I_{G\boxtimes H}[S_1 \times S_2]$. □

Corollary 1. *Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ for graphs G and H. For every integer $r \geq 0$,*

$$I_G^r[S_1] \times I_H^r[S_2] \subseteq I_{G\boxtimes H}^r[S_1 \times S_2].$$

Proof. We proceed by induction on r. It is trivial for $r = 0$. Assume now that $r \geq 1$. By the induction hypothesis, $I_G^{r-1}[S_1] \times I_H^{r-1}[S_2] \subseteq I_{G\boxtimes H}^{r-1}[S_1 \times S_2]$. Hence, by lemma 3,

$$\begin{aligned} I_G^r[S_1] \times I_H^r[S_2] &= I_G[I_G^{r-1}[S_1]] \times I_H[I_H^{r-1}[S_2]] \\ &\subseteq I_{G\boxtimes H}[I_G^{r-1}[S_1] \times I_H^{r-1}[S_2]] \\ &\subseteq I_{G\boxtimes H}[I_{G\boxtimes H}^{r-1}[S_1 \times S_2]] = I_{G\boxtimes H}^r[S_1 \times S_2], \end{aligned}$$

as desired. □

As a direct consequence of this corollary, we obtain the following result.

Proposition 1. *Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ for graphs G and H. If S_1 is a geodetic (resp. hull) set of G and S_2 is a geodetic (resp. hull) set of H, then $S_1 \times S_2$ is a geodetic (resp. hull) set of $G \boxtimes H$.*

Proof. Let r, s be positive integers such that $I_G^r[S_1] = V(G)$ and $I_H^s[S_2] = V(H)$. We may assume that $r \leq s$. Then, $V(G \boxtimes H) = V(G) \times V(H) = I_G^s[S_1] \times I_H^s[S_2] \subseteq I_{G\boxtimes H}^s[S_1 \times S_2]$. □

Proposition 2. *Let $S \subseteq V(G \boxtimes H)$ for graphs G and H. If S is a geodetic set of $G \boxtimes H$, then either the projection $\mathrm{Pr}_G(S)$ of S onto G or the projection $\mathrm{Pr}_H(S)$ of S onto H is geodetic.*

Proof. Assume that neither $S_1 = p_G(S)$ nor $S_2 = p_H(S)$ is geodetic and consider $g \in V(G) \setminus I_G[S_1]$ and $h \in V(H) \setminus I_H[S_2]$. Since $(g,h) \in I_{G\boxtimes H}[S] = V(G \boxtimes H)$, then $(g,h) \in I_{G\boxtimes H}[(g',h'),(g'',h'')]$ for some $(g',h'),(g'',h'') \in S$. Hence,

$$d_{G\boxtimes H}((g',h'),(g'',h'')) = d_{G\boxtimes H}((g',h'),(g,h)) + d_{G\boxtimes H}((g,h),(g'',h'')).$$

On the other hand, as $g \notin I[g',g'']$ and $h \notin I[h',h'']$, we have that $d_G(g',g'') < d_G(g',g) + d_G(g,g'')$ and $d_H(h',h'') < d_H(h',h) + d_H(h,h'')$. Hence,

$$\begin{aligned} \max\{d_G(g',g''), d_H(h',h'')\} &< \max\{d_G(g',g) + d_G(g,g''), d_H(h',h) + d_H(h,h'')\} \\ &\leq \max\{d_G(g',g), d_H(h',h)\} + \max\{d_G(g,g''), d_H(h,h'')\} \\ &= d_{G\boxtimes H}((g',h'),(g,h)) + d_{G\boxtimes H}((g,h),(g'',h'')) \end{aligned}$$

which contradicts the previous expression for the distance between (g',h') and (g'',h''). □

This property is far for being true for hull sets. For example, let C_5 and C_7 be the cycles of order 5 and 7 respectively with vertices $V(C_5) = \{0,1,2,3,4\}$, $V(C_7) = \{0,1,2,3,4,5,6,\}$, and edges $E(C_5) = \{01,12,23,34,41\}$, $E(C_7) = \{01,12,23,34,45,56,60\}$. It is easy to prove that $\{(0,0),(1,3)\}$ is a hull set of $C_5 \boxtimes C_7$, but $\{0,1\}$ is not a hull set of C_5 and $\{0,3\}$ is not a hull set of C_7.

Proposition 3. *Let G and H be two graphs such that G has no simplicial vertices. If S is a hull set of G and x is an arbitrary vertex of H, then $S \times \{x\}$ is a hull set of $G \boxtimes H$.*

Proof. We prove by induction on $k \geq 0$ that for every vertex y of H such that $d_H(x,y) = k \geq 0$, then for every $u \in G$ the vertex (u,y) is in the convex hull of $S \times \{x\}$. For $k = 0$, the condition $d_H(x,y) = k \geq 0$ implies $y = x$. Since S is a hull set of G, for every u in G we have $u \in I^n[S]$ for some $n \geq 0$. By corollary 1, $(u,x) \in I^n[S] \times I^n[\{x\}] = I^n[S \times \{x\}]$, and consequently, (u,x) is in the convex hull of $S \times \{x\}$.

Suppose now $k > 0$ and consider a vertex y with $d_H(x,y) = k > 0$. Let z be a vertex in H adjacent to y and at distance $k-1$ from x. Since G has no simplicial vertices, for every vertex u in G there exist vertices v,w in G adjacent to u such that $d_G(v,w) = 2$. Thus $d_{G\boxtimes H}((v,z),(w,z)) = 2$, $(u,y)(v,z) \in E(G \boxtimes H)$ and $(u,y)(w,z) \in E(G \boxtimes H)$, that is, (u,y) is in a shortest path between (v,z) and (w,z). By induction hypotheses, (v,z) and (w,z) are in the convex hull of $S \times \{x\}$. Therefore (u,y) is in the convex hull of $S \times \{x\}$. □

4 Bounds of the geodetic and hull numbers

Obviously, the hull number of a non trivial graph is at least 2. The following result gives us the minimum value that can attain the hull number of the strong product of two non trivial graphs. Although the result is very simple, the proof is long and is omitted because of lack of space, nevertheless it can be done by tedious case analysis.

Proposition 4. *Let G and H be nontrivial graphs. Then, $g(G \boxtimes H) \geq 4$.*

Proof. (Omitted.)

As a direct consequence of propositions 1 and 2, we obtain bounds for the geodetic number of the strong product of two graphs in terms of the geodetic numbers of its factor graphs.

Theorem 1. *For any two graphs G and H,*

$$\min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H),$$

and both bounds are sharp.

Proof. First, we prove the upper bound. Let S_1 and S_2 be geodetic sets of G and H with minimum cardinality, that is, such that $|S_1| = g(G)$ and $|S_2| = g(H)$. By Proposition 1, $S_1 \times S_2$ is a geodetic set of $G \boxtimes H$ with cardinality $|S_1 \times S_2| = |S_1||S_2| = g(S_1)g(S_2)$. Hence, $g(G \boxtimes H) \leq g(G)g(H)$.

To prove the lower bound, take a minimum geodetic set S of $G \boxtimes H$. According to Proposition 2, we may suppose, without loss of generality, that $p_G(S)$ is a geodetic set of G. Hence,

$$\min\{g(G), g(H)\} \leq g(G) \leq |p_G(S)| \leq |S| = g(G \boxtimes H).$$

To show the sharpness of the upper bound, take $G = K_m$ and $H = K_n$. Then, $g(K_m \boxtimes K_n) = g(K_{mn}) = mn = g(K_m)g(K_n)$. Finally, to show the sharpness of the lower bound, take $G = K_{r,s}$ a complete bipartite graph and $H = K_n$, with $r, s, n \geq 4$. Then, $g(K_{r,s} \boxtimes K_n) = 4 = \min\{g(K_{r,s}), g(K_n)\}$. □

Theorem 2. *For any two graphs G and H, we have*

$$h(G \boxtimes H) \leq h(G)h(H)$$

and this bound is sharp.

Proof. Let S_1 and S_2 be hull sets of G and H with minimum cardinality, that is, such that $|S_1| = h(G)$ and $|S_2| = h(H)$. By Proposition 1, $S_1 \times S_2$ is a hull set of $G \boxtimes H$ with cardinality $|S_1 \times S_2| = |S_1||S_2| = h(S_1)h(S_2)$. Hence, $h(G \boxtimes H) \leq h(G)h(H)$.

To prove the sharpness of this bound, take $G = K_m$ and $H = K_n$ and notice that $h(K_m \boxtimes K_n) = h(K_{mn}) = mn = h(K_m)h(K_n)$ □

As a particular case, if a graph has no simplicial vertices, proposition 3 gives the following upper bounds.

Theorem 3. *Let G and H be two graphs such that G has no simplicial vertices, then*

$$h(G \boxtimes H) \leq h(G).$$

Corollary 2. *Let G and H be two graphs with no simplicial vertices, then*

$$h(G \boxtimes H) \leq \min\{h(G), h(H)\}.$$

5 Strong products of paths, cycles and complete graphs

Let P_n, C_n and K_n denote respectively the path, cycle and complete graph of order n. We determine in this section the values of the geodetic and hull number of these graphs.

First of all we consider a special class of graphs. A graph G is called *extreme geodesic* if the set of its simplicial vertices is geodesic (see [9]). Since $Ext(G)$

is included in every geodetic and hull set, $Ext(G)$ is the unique minimum geodetic set and also the unique minimum hull set of extreme geodesic graphs and, therefore, $h(G) = g(G) = |Ext(G)|$. Trees and complete graphs are examples of extreme geodesic graphs. On the other hand, observe that a vertex (u, v) is a simplicial vertex in $G \boxtimes H$ if and only if both u and v are simplicial vertices in G and H, respectively, i.e., $Ext(G) \times Ext(H) = Ext(G \boxtimes H)$. As a direct consequence of this equality and Proposition 1, we have that two graphs G and H are extreme geodesic if and only if $G \boxtimes H$ is extreme geodesic, which means that $h(G \boxtimes H) = g(G \boxtimes H) = |Ext(G \boxtimes H)| = |Ext(G) \times Ext(H)| = |Ext(G)| \cdot |Ext(H)| = g(G)g(H) = h(G)h(H)$. As a direct consequence of these facts, we obtain the geodetic and hull numbers of strong products of complete graphs, trees and paths (as a particular case of trees). The results are shown in Table 2.

$G \diagdown H$	P_n	T_n^k	K_n
P_m	4	$2k$	$2n$
T_m^h	$2h$	hk	hn
K_m	$2m$	mk	mn

Table 2. Geodetic and hull number of some strong products. P_m denotes the path of order m, T_m^h an arbitrary tree with n vertices and h leaves and K_m the clique of order m.

We give now some results involving complete graphs.

Lemma 4. *Let G be a graph with a geodetic set S satisfying the following condition:*

$$\forall x \in S, \exists y, z \in S \setminus \{x\} \text{ such that } x \in I[y, z]. \tag{1}$$

Then, for every vertex $k \in V(K_n)$, $S \times \{k\}$ is a geodetic set of $g(G \boxtimes K_n)$.

Proof. Take an arbitrary vertex $(v, h) \in V(G \boxtimes K_n)$. This means that there exist a pair of vertices $s, s' \in S \setminus \{v\}$ such that $v \in I[s, s']$. Hence,

$$\begin{aligned} d((s,k),(s',k)) &= d(s,s') = d(s,v) + d(v,s') \\ &= d((s,k),(v,h)) + d((v,h),(s',k)). \end{aligned}$$

Hence, $(v, h) \in I[(s, k), (s', k)] \subset I[S]$, as desired. □

Proposition 5. *Let G be a graph with a minimum geodetic set S satisfying condition* (1)*. Then, for every positive integer n, $g(G \boxtimes K_n) = g(G)$.*

Proof. As a corollary of Lemma 4 we have that $g(G \boxtimes K_n) \le g(G)$. To get the equality, suppose that $R = \{(g_1, k_1), (g_2, k_2), \ldots, (g_m, k_m)\}$ is a geodetic set in $G \boxtimes K_n$ such that $m = |R| < |S| = g(G)$. Consider the set $R' =$

$\{(g_1,k_1),(g_2,k_1),\dots,(g_m,k_1)\}$. For every vertex $(g,k)\in G\boxtimes K_n$ we have that $(g,h)\in I[(g_i,k_i),(g_j,k_j)]$ for some $i,j\in\{1,\dots,m\}$. Hence, $g\neq g_i\neq g_j\neq g$ and

$$\begin{aligned} d((g_i,k_1),(g_j,k_1)) &= d(g_i,g_j) = d((g_i,k_i),(g_j,k_j)) \\ &= d((g_i,k_i),(g,k)) + d((g,k),(g_j,k_j)) \\ &= d(g_i,g) + (g,g_j) \\ &= d((g_i,k_1),(g,k)) + d((g,k),(g_j,k_1)). \end{aligned}$$

In other words, $(g,k)\in I[(g_i,k_1),(g_j,k_1)]\subseteq I[R']$. We have thus proved that R' is also a geodetic set of $G\boxtimes K_n$. Furthermore, as a direct consequence of Proposition 2, we conclude that the projection $p_G(R')$ is a geodetic set of G, implying that $|p_G(R')|=|R'|\le|R|<|S|=g(G)$, a contradiction. □

Proposition 6. *Let $n\ge 4$ be an even integer and let G be a graph of order $m\ge 2$. If $m\ge 2$, then*

$$g(P_m\boxtimes C_n)=g(K_m\boxtimes C_n)=4,$$
$$h(P_m\boxtimes C_n)=h(K_m\boxtimes C_n)=2.$$

Proof. The equality $h(G\boxtimes C_n)=2$ is a direct consequence of Theorem 3. The equality $g(P_m\boxtimes C_n)=4$ is a corollary of both Proposition 4 and the upperbound displayed in Theorem 1. Finally, to prove that $g(K_m\boxtimes C_n)=4$ it is enough to consider again Proposition 4 and to notice that if $V(C_n)=\{u_1,u_2,\dots,u_n\}$, then the set $S=\{u_1,u_2,u_{\frac{n}{2}+1},u_{\frac{n}{2}+2}\}$ is a (not minimum) geodetic set in C_m satisfying the condition (1) pointed out in Lemma 4. □

Proposition 7. *Let $n\ge 5$ be an odd integer. If $m\ge 2$, then*

$$g(K_m\boxtimes C_n)=5,$$
$$h(K_m\boxtimes C_n)=3.$$

Proof. Notice that if $n=2k+1$ and $V(C_n)=\{c_0,c_1,\dots,c_{2k}\}$, then $S=\{c_0,c_1,c_k,c_{k+1},c_{k+2}\}$ is a (not minimum) geodetic set of C_n satisfying the condition (1) pointed out in Lemma 4, which allows us to conclude that $g(K_m\boxtimes C_n)\le 5$. To get the equality, suppose that there exist a geodetic set $R=\{(j_1,c_{i_1}),(j_2,c_{i_2}),(j_3,c_{i_3}),(j_4,c_{i_4})\}$ of cardinality 4 in $K_m\boxtimes C_n$. In particular, $\{c_{i_1},c_{i_2},c_{i_3},c_{i_4}\}$ is a set of vertices of C_n of cardinality at least 2, since otherwise R is a convex set of $K_m\boxtimes C_n$. Note that n is odd, so we may suppose that c_{i_1} is not in any shortest path between two vertices belonging to $\{c_{i_2},c_{i_3},c_{i_4}\}$, which means that $(K_m\times\{c_i\})\cap(I[R]\setminus R)=\emptyset$, a contradiction.

To prove that $h(K_m\boxtimes C_n)=3$ it suffices to show that $h(K_m\boxtimes C_n)>2$, since according to Theorem 3, $h(K_m\boxtimes C_n)\le h(C_n)=3$. To this end, take an arbitrary set $R=\{(j_1,c_{i_1}),(j_2,c_{i_2})\}$ of cardinality 2 in $K_m\boxtimes C_n$. If $c_{i_1}=c_{i_2}$, then $CH[R] = R$, i.e., in this case R is not a hull set of $h(K_m \boxtimes C_n)$. Assume thus that $c_{i_1} \neq c_{i_2}$, and w.l.o.g. that $R = \{(1, c_0), (j, c_h)\}$, where $j \in \{1, 2\}$, $n = 2k + 1$ and $0 < h \leq k$. It is straightforward to verify that $CH[R] \setminus R = \bigcup_{i=1}^{h-1}(K_m \times \{c_i\})$, which means that neither in this case R is a hull set of $h(K_m \boxtimes C_n)$. □

Finally, we give the possible values for the geodetic and hull numbers of the strong product of cycles by paths or cycles. The proofs of these results are not completely included because of lack of space. We give only a sketch of the proof.

Proposition 8. *Let $n \geq 5$ be an odd integer. If $m \geq 2$, then*

$$5 \leq g(P_m \boxtimes C_n) \leq 6,$$
$$2 \leq h(P_m \boxtimes C_n) \leq 3,$$

Proof. By Proposition 4 and Theorem 1 we have $4 \leq g(P_m \boxtimes C_n) \leq 6$. It remains to prove that $g(P_m \boxtimes C_n) \neq 4$. For this, we consider a set S of at most 4 diferent vertices of the graph and find in any case a vertex that is not in $I[S]$ using Lemma 2. We omit details here. The lower bound of the hull number is trivial, and the upper bound is consequence of Corollary 2.

Proposition 9. *Let $m, n \geq 4$ be two integers. Then,*

$$4 \leq g(C_m \boxtimes C_n) \leq 7,$$
$$2 \leq h(C_m \boxtimes C_n) \leq 3.$$

Moreover,

$$g(C_m \boxtimes C_n) = 4 \text{ if } m, n \text{ are even},$$
$$h(C_m \boxtimes C_n) = 3 \text{ if and only if } m = n \text{ odd}.$$

Proof. The lower bound of the geodetic number is consequence of Proposition 4. If m and n are even, consider two vertices u, v in C_m at distance $m/2$ and two vertices a, b in C_n at distance $n/2$. It can be proved that $\{(u, a), (u, b), (v, a), (v, b)\}$ is a geodetic set. In the remaining cases, we give a geodetic set of cardinality 7. The lower bound of the hull number is trivial, and the upper bound is consequence of Corollary 2. If m is even and u, v are vertices of C_m at distance $m/2$, then Proposition 3 implies that $\{(u, a), (v, a)\}$ is a hull set for any vertex a in C_n, that is, the hull number is 2. Similarly, the hull number is 2 if n is even. For m, n odd integers such that $m \neq n$, suppose that $m = 2k + 1 > 2h + 1 = n$. Take two vertices u, v in C_m at distance $h + 1$ and two vertices a, b in C_n at distance h. Using Proposition 3 we get that $\{(u, a), (v, b)\}$ is a hull set, and hence, the hull number is 2. Finally, it remains to see that for any 2 vertices of a product of odd cycles, there is a vertex of the graph not lying in its convex hull. For this, we study the shape of the geodetic interval in these graphs.

We summarize these results in the following table.

	geodetic number	**hull number**
$P_m \boxtimes C_n$	$\begin{cases} 4, & \text{if } n \text{ is even;} \\ 5 \text{ or } 6, & \text{if } n \text{ is odd.} \end{cases}$	$\begin{cases} 2, & \text{if } n \text{ is even;} \\ 2 \text{ or } 3, & \text{if } n \text{ is odd.} \end{cases}$
$K_m \boxtimes C_n$	$\begin{cases} 4, & \text{if } n \text{ is even;} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$	$\begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$
$C_m \boxtimes C_n$	$\begin{cases} 4, & \text{if } m \text{ and } n \text{ are even;} \\ 4, 5 \text{ or } 6, & \text{if } m \text{ is even and } n \text{ is odd;} \\ 4, 5, 6 \text{ or } 7, & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$	$\begin{cases} 3, \text{if } m = n \text{ odd;} \\ 2, \text{ otherwise.} \end{cases}$

Table 3. Geodetic and hull numbers of some strong product graphs of the form $G \boxtimes C_n$.

References

[1] B. Bresar, S. Klavzar and A. Tepeh Horvat. On the geodetic number and related metric sets in Cartesian product graphs. *Discrete Mathematics*, 308(23):5555–5561, 2008.

[2] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas and C. Seara. On geodetic sets formed by boundary vertices. *Discrete Mathematics*, 306:188–198, 2006.

[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas and C. Seara. Geodeticity of the contour of chordal graphs *Discrete Applied Math.*, 156(7):1132–1142, 2008.

[4] J. Cáceres, A. Márquez, O. R. Oellermann and M. L. Puertas. Rebuilding covex sets in graphs. *Discrete Mathematics*, 297:26–37, 2005.

[5] G. B. Cagaanan and S. R. Canoy. On the hull sets and hull number of the cartesian product of graphs *Discrete Mathematics*, 287(1–3):141–144, 2004.

[6] S. R. Canoy and G. B. Cagaanan. On the hull number of the composition of graphs. *Ars Combinatoria*, 75:113–119, 2005.

[7] S. R. Canoy, G. B. Cagaanan and S. V. Gervacio. Convexity, geodetic and hull numbers of the join of graphs. *Utilitas Mathematica*, 71:143–159, 2006.

[8] G. Chartrand, D. Erwin, G. L. Jonhs and P. Zhang. Boundary vertices in graphs. *Discrete Mathematics*, 263:25–34, 2003.

[9] G. Chartrand and P. Zhang. Extreme Geodesic Graphs. *Czechoslovak Mathematical Journal*, 52(4):771–780, 2004.

[10] M. G. Everett and S. B. Seidman. The hull number of a graph. *Discrete Mathematics*, 57(3):217–223, 1985.

[11] M. Farber and R. E. Jamison. Convexity in graphs and hypergraphs. *SIAM J. Alg. Disc. Meth.*, 7:433-444, 1986.

[12] F. Harary, E. Loukakis and C. Tsouros. The geodetic number of a graph. *Mathematical and Computer Modelling*, 17(11):89–95, 1993.

[13] W. Imrich and S. Klavzar. *Product graphs: Structure and Recognition.* Wiley-Interscience, New York, 2000.

[14] T. Jiang, I. M. Pelayo and D. Pritikin. Geodesic convexity and cartesian products in graphs. Manuscrit.

[15] I. M. Pelayo. Generalizing the Krein-Milman property in graph convexity spaces: a short survey. *Ramanujan Mathematical Society Lecture Notes Series*, 5:131–142, 2008.

[16] D. B. West. *Introduction to Graph Theory.* Prentice-Hall, New Jersey, 1999.