On the Metric Dimension of Infinite Graphs[1]

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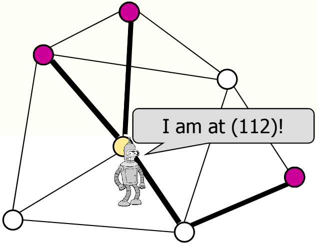
Locating sets, Reference sets, Location number

P. Slater, Leaves of trees. Congressuss Numerantium, 14 (1975) 549-559.



Resolvings sets, Metric basis, Metric dimension

F. Harary and R. Melter, On the metric dimension of a graph. Ars Combinatoria, 2 (1976) 191-195.

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- Let $G = (V, E)$ be a connected graph and $v, w \in V$.

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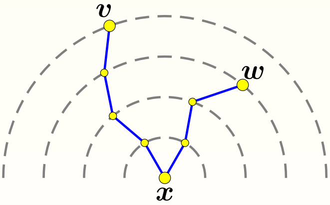

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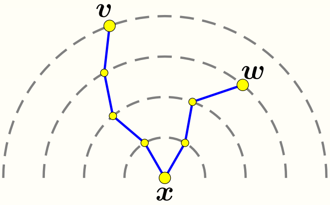

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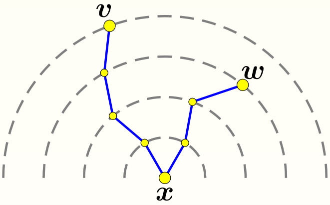

- $S \subseteq V$ is called a *resolving set* of G if every pair of distinct vertices $v, w \in V$ are resolved by some vertex $x \in S$.
- Let $S = \{u_1, \ldots, u_k\}$ be a resolving set. The ordered set $(d(x, u_1), \ldots, d(x, u_k))$ are said to be the *metric coordinates* of $x \in V$ with respect to S.

Metric Basis and Metric Dimension

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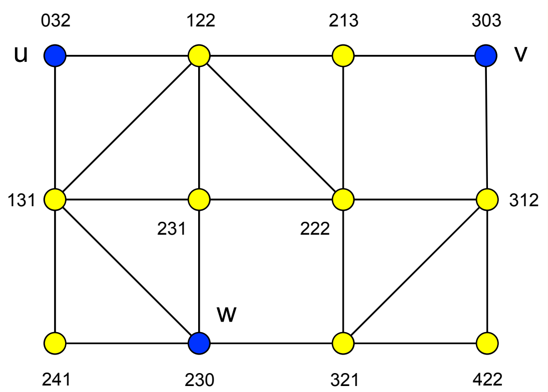

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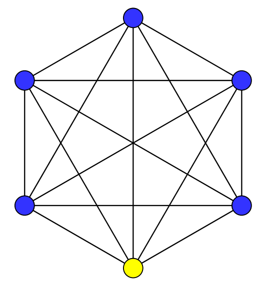

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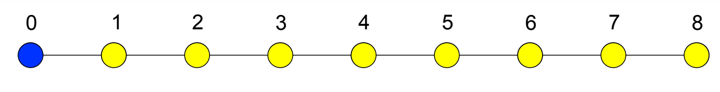

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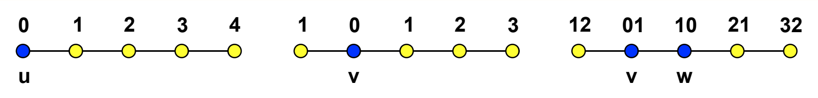

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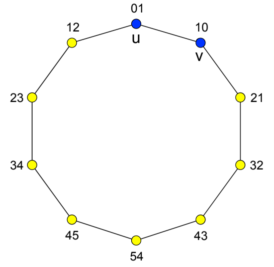
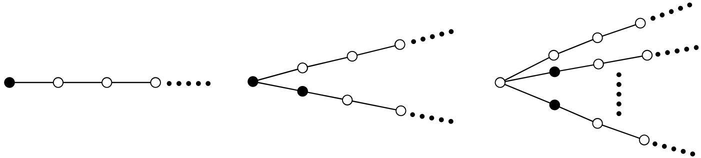
- ▷ $\beta(G) = 1$ if and only if $G = P_n$ (finite case)
- ▷ There are resolving sets not containing a metric basis
- ▷ $\beta(C_n) = 2$ (Metric basis: two no antipodal vertices)



Infinite Graphs (some exmaples)

- The infinite path, P_∞, and the k-infinite path, $P_{k\infty}$



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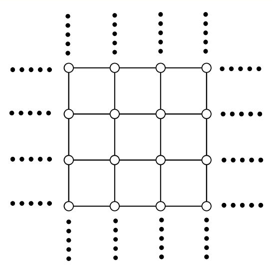

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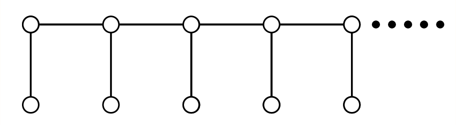

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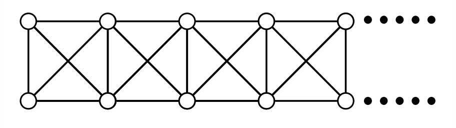

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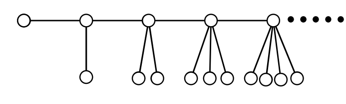

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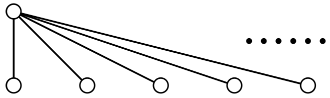
- The infinite star $S_{1\infty}$



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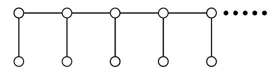
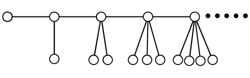
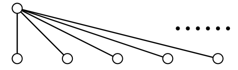
- *Locally finite graph*: $\forall u \in V$, $\deg(u) < \infty$
- *Uniformly locally finite graph (ULF)*:
  there exists $K \geq 0$ such that $\forall u \in V$, $\deg(u) \leq K$

Locally Finite Infinite Graphs

Proposition [König, 1936]

G connected locally finite infinite graph $\Rightarrow$ V is countable

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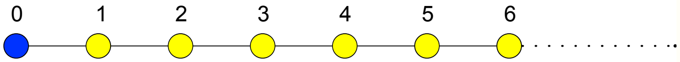
All infinite graphs considered in this work are locally finite

Example: P_∞

- $\beta(P_\infty) = 1$
- Metric basis: the vertex of degree 1



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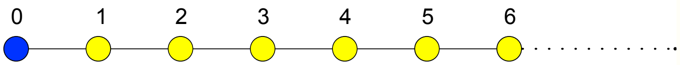
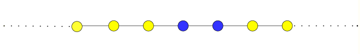


Proposition

If G is a connected graph, $\beta(G) = 1 \Leftrightarrow G$ is a finite path P_n or the infinite path P_∞

Example: $P_{k\infty}$

- $\beta(P_{2\infty}) = 2$
- Metric basis: any two vertices of the graph



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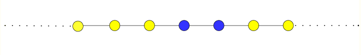
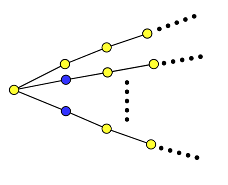


- $\beta(P_{k\infty}) = k - 1$, if $k \geq 3$
- Metric basis: $k - 1$ vertices of distinct connected components of $G \setminus u$, where u is the vertex of degree k

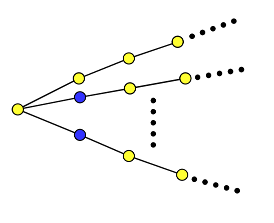


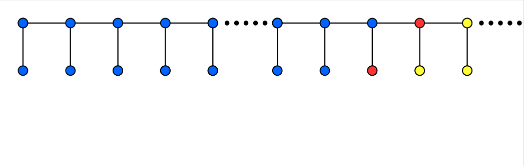
Outline

Existence of Infinite Graphs with finite/infinite Metric Dimension

There are infinite graphs

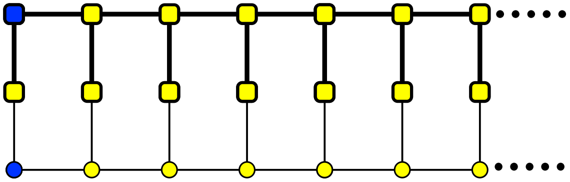
- with metric dimension k, for any $k \geq 1$,
  for example, the graphs P_∞ and $P_{(k+1)\infty}$, $k \geq 2$
- with infinite metric dimension,
  for example, the *infinite comb*





Subgraphs and Metric Dimension

- There are infinite graphs with finite metric dimension containing induced subgraphs with infinite metric dimension



Blue vertices: metric basis

Squared vertices: induce a subgraph with infinite metric dimension

Finite Metric Dimension implies ULF

- G connected infinite graph with $\beta(G) = n \Longrightarrow$ the degree of all vertices of G is at most $3^n - 1$.
- Finite metric dimension $\Longrightarrow$ ULF

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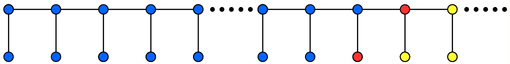
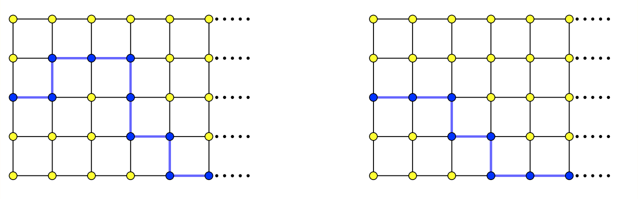
Remark

The reciprocal is not true: the infinite comb is a counterexample



Metric Rays

- A *metric ray with endpoint* $u_0 \in V$ is an infinite set of vertices $\{u_0, u_1, \ldots, u_k, \ldots\}$ such that $u_i u_{i+1} \in E$ and $d(u_0, u_i) = i$ for all $i \geq 0$



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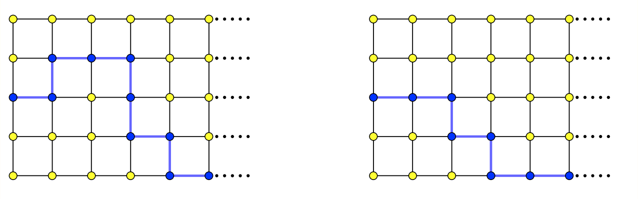

Lemma [König 1936]

For every vertex v of an infinite connected graph, there exists a metric ray with v as endpoint.

Metric Rays

- A connected infinite graph with finite metric dimension does not contain an infinite number of pairwise disjoint metric rays.

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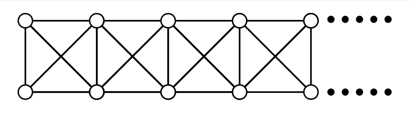
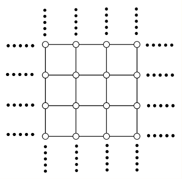


The conditions

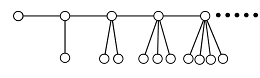
(1) uniformly locally finite

(2) not containing infinite pairwise disjoint metric rays

are independent and not sufficient to assure finite metric dimension.



(1), no **(2)**



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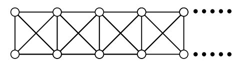

(1), **(2)**

$\beta(G) = \infty$

Outline

Infinite Trees

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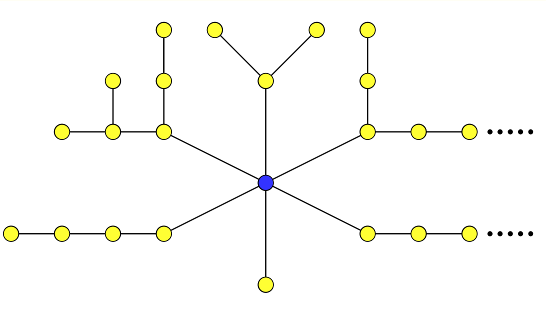
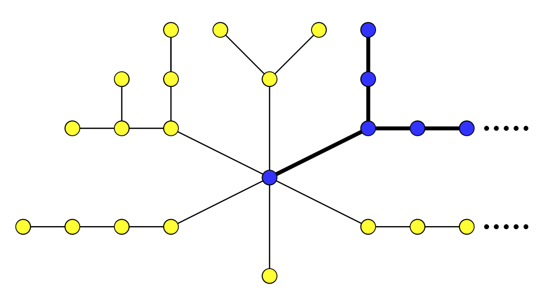
- *Leg* of a tree T at vertex v: maximal subtree of T containing v as a leaf such that is a finite or an infinite path
- $L_T(v) = \#$ legs of T at v

Example: Legs of T at v



Example: Legs of T at v



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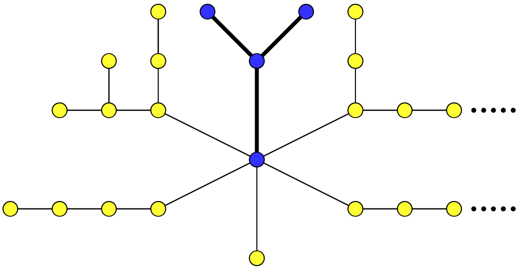

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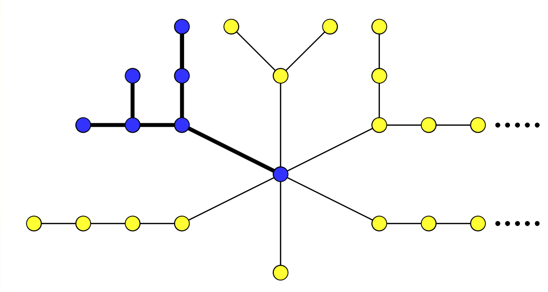

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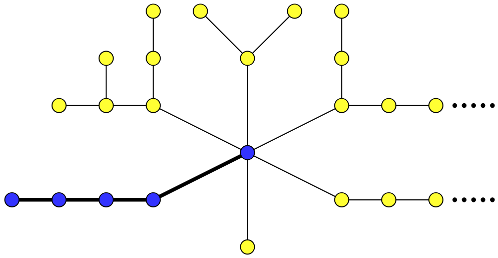

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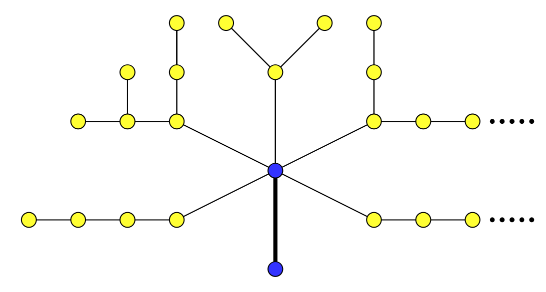

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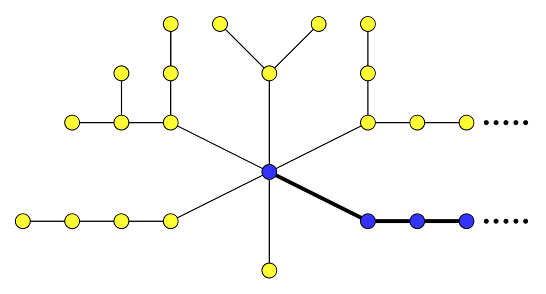
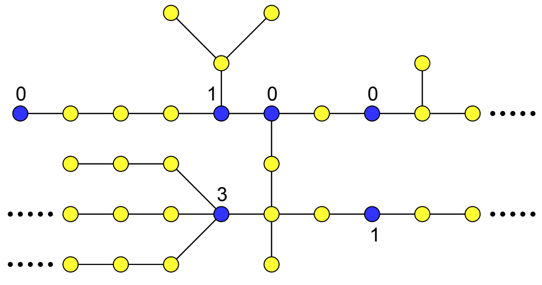


Legs of T at v

- $L_T(v) \leq \deg_T(v)$
- $L_T(v) = 0$, if $\deg_T(v) = 1$ and $T \neq P_n, P_\infty$
- $L_T(v) \leq 1$, if $\deg_T(v) = 2$ and $T \neq P_n, P_\infty, P_{2\infty}$



The parameter $L_T(v)$

Infinite Trees with Finite Metric Dimension

- An infinite tree has finite metric dimension $\Longleftrightarrow$ the set of vertices of degree at least three is finite.

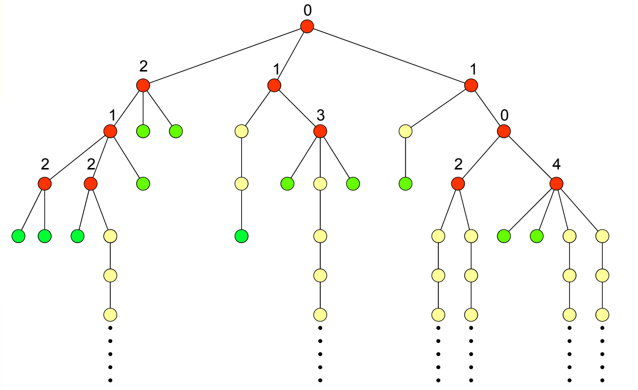
Infinite Trees with Finite Metric Dimension

- An infinite tree has finite metric dimension $\iff$ the set of vertices of degree at least three is finite.
- $T \neq P_n, P_\infty, P_{2\infty}$ tree with finite metric dimension, then

$$\beta(T) = \sum_{v \in W} \max\{L_T(v) - 1, 0\},$$

where $W = \{v \in V(T) : d_T(v) \geq 3\}$

Example: Metric Dimension of an Infinite Tree

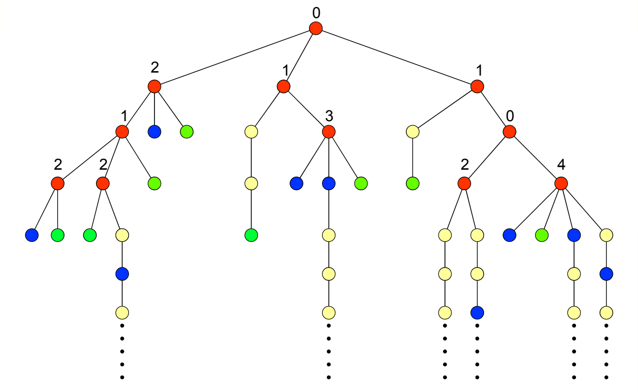


The parameter L_T for the vertices of degree ≥ 3

$$\beta(T) = (4-1) + (3-1) + 4\,(2-1) + 3\,(1-1) + 2 \cdot 0 = 9$$

Metric Basis of Infinite Trees

- *Metric basis:* for each $v \in W$ such that $L_T(v) = k \geq 2$, select $k - 1$ vertices $\neq v$ of distinct legs of T at v

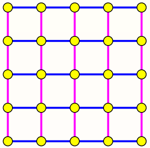


Blue vertices: metric basis of T

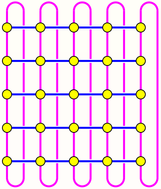
Outline

Cartesian Products

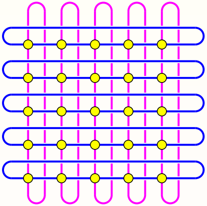
- Cartesian product $G \Box H$ of graphs G, H:
  $V(G \Box H) = V(G) \times V(H)$
  $$(a, v) \sim (b, w) \Leftrightarrow \begin{cases} a = b \text{ and } vw \in E(H), \text{ or} \\ ab \in E(G) \text{ and } v = w \end{cases}$$



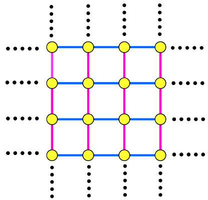
$P_5 \Box P_5$



$P_5 \Box C_5$



$C_5 \Box C_5$



$P_{2\infty} \Box P_{2\infty}$

Projections

- *Projection* of $S \subseteq V(G \square H)$ onto G:
  $\{a \in V(G) | (a, v) \in S \text{ for some } v \in V(H)\}$
- *Projection* of $S \subseteq V(G \square H)$ onto H:
  $\{v \in V(H) | (a, v) \in S \text{ for some } a \in V(G)\}$

Projections

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- *Projection* of $S \subseteq V(G\Box H)$ onto H:
  $\{v \in V(H) | (a, v) \in S \text{ for some } a \in V(G)\}$

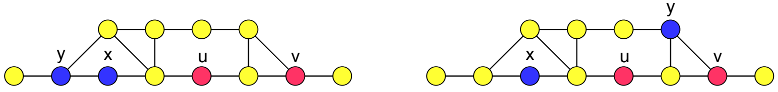
- S resolving set of $G\Box H \Rightarrow$
  the projection of S onto G is a resolving set of G and
  the projection of S onto H is a resolving set of H

- $\beta(G\Box H) \geq \max\{\beta(G), \beta(H)\}$

Doubly Resolving Sets

[Cáceres et al. 2007]

- Two vertices x and y in a connected graph G with at least two vertices (finite or infinite) *doubly resolve* a pair of vertices u and v if $d(u,x) - d(v,x) \neq d(u,y) - d(v,y)$



$2 - 4 = 3 - 5 = -2$ x, y do not doubly resolve u, v

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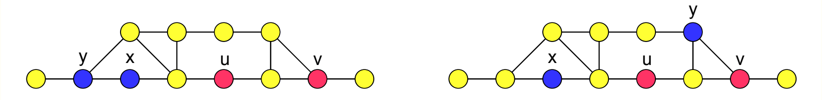

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$2 - 4 \neq 2 - 1$ x, y double resolve u, v

- $S \subseteq V$ *doubly resolves* $U \subseteq V$ if every pair of distinct vertices in U are doubly resolved by two vertices in S.

Doubly Resolving Sets

- S is a *doubly resolving set* of G if S doubly resolves $V(G)$.
- If $G \neq K_1$ has at least a finite doubly resolving set, we define $\psi(G)$ as the minimum cardinality of a doubly resolving set. Otherwise, we say that $\psi(G) = \infty$.

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- $\psi(P_n) = 2$, $n \geq 2$
- $\psi(C_n) = 2$, if n is odd, and $\psi(C_n) = 3$, if n is even
- $\psi(K_n) = n - 1$, if $n \geq 3$

Doubly Resolving Sets

- A finite set of vertices does not doubly resolve an infinite set of vertices of a connected infinite graph.
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- G and H connected infinite graphs $\Longrightarrow \beta(G\Box H) = \infty$.
- G connected infinite graph with infinite metric dimension $\Longrightarrow$ $\beta(G\Box H) = \infty$.
- G connected infinite graph with finite metric dimension and H finite connected graph with at least two vertices $\Longrightarrow \beta(G\Box H) < \infty$ and $\beta(G\Box H) \leq \beta(G) + \psi(H) - 1$

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$\lvert V(G) \rvert$	$\lvert V(H) \rvert$	$\beta(G \square H)$
$< \infty$	$< \infty$	$< \infty$
$= \infty$	$= \infty$	$= \infty$
$= \infty$	$< \infty$	$\begin{cases} < \infty & \text{if } \beta(G) < \infty, \\ = \infty & \text{if } \beta(G) = \infty. \end{cases}$

Metric Dimension of $P_{\infty}\Box H$ and $P_{2\infty}\Box H$

- $\beta(P_{\infty}\Box H) \leq \psi(H)$
- $\beta(P_{2\infty}\Box H) \leq \psi(H) + 1$

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- If H is a connected graph, then $\beta(P_{2\infty} \Box H) = 2 \iff H$ is the trivial graph, K_1
- H connected graph and $S \subseteq \{0\} \times V(H)$ resolving set of $P_\infty \Box H \implies$ for any $u \in V(H)$, $S' = S \cup \{(1, u)\}$ is a resolving set of $P_{2\infty} \Box H$.

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$G \backslash H$	P_n, $n \geq 2$	C_n, $n \geq 3$ odd	C_n, $n \geq 4$ even	K_n, $n \geq 4$
P_∞	2	2	3	$n-1$
$P_{2\infty}$	3	3	4	$n-1$

Open Problems

- Characterize infinite graphs with finite metric dimension
- Characterize ULF graphs with finite metric dimension
- Determine the metric dimension of some families of infinite graphs