On locating and dominating sets in graphs[1]

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Summary

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$G = (V, E)$ is a simple connected graph.

Definition

A set D of vertices in G is a *dominating set* if, for every $u \in V(G) \setminus D$:

$$N(u) \cap D \neq \emptyset$$

Definition

The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Definition

A set $D = \{x_1, \ldots, x_k\}$ is a *locating set* if, for every pair $u, v \in V(G)$,

$$(d(u, x_1), \ldots, d(u, x_k)) \neq (d(v, x_1), \ldots, d(v, x_k)).$$

Definition

The *metric dimension* (also called the *location number*) $\beta(G)$ is the minimum cardinality of a locating set of G.

Summary

$G = (V, E)$ is a simple connected graph.

Definition

A set D of vertices in a graph G is a *locating dominating set* if it is both locating and dominating.

Definition

The *metric-location-domination number* $\eta(G)$ is the minimum cardinality of a locating dominating set of G.

Proposition

Let $S_1, S_2 \subseteq V(G)$. If S_1 is dominating and S_2 is locating, then $S_1 \cup S_2$ is both locating and dominating. Hence,

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)$$

| $\gamma(G) = 2$ | $\beta(G) = 3$ | $\eta(G) = 4$ |
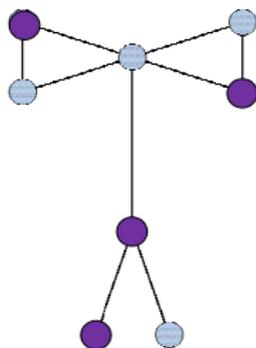
In this example: $\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 4 \leq \gamma(G) + \beta(G) = 5$

Summary

$G = (V, E)$ is a simple connected graph.

Definition

A set D of vertices in a graph G is a *locating-dominating set* if for every two vertices $u, v \in V(G) \setminus D$, $\emptyset \neq N(u) \cap D \neq N(v) \cap D \neq \emptyset$.
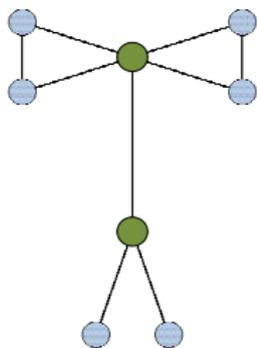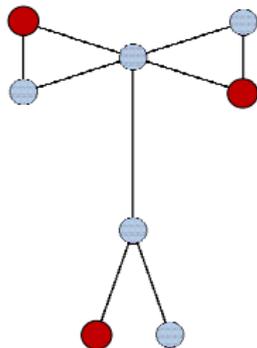
Definition

The *location-domination number* $\lambda(G)$ is the minimum cardinality of a locating-dominating set of G.

Proposition

Every locating-dominating set is both locating and dominating. Hence,

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \lambda(G)$$

| $\beta(G) = 2$ | $\gamma(G) = \eta(G) = 3$ | $\lambda(G) = 4$ |

In this example:

$$\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4$$

Summary

G	γ	β	η	λ
P_2	1	1	1	1
P_3	1	1	2	2
$P_n, n > 3$	$\lceil \frac{n}{3} \rceil$	1	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$
C_4, C_5	2	2	2	2
C_6	2	2	3	3
$C_n, n > 6$	$\lceil \frac{n}{3} \rceil$	2	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$
$K_n, n > 1$	1	$n-1$	$n-1$	$n-1$
$K_{1,n-1}, n > 2$	1	$n-2$	$n-1$	$n-1$
$K_{r,n-r}, r > 1, n > 4$	2	$n-2$	$n-2$	$n-2$
$W_{1,4}$	1	2	2	2
$W_{1,5}$	1	2	3	3
$W_{1,6}$	1	3	3	3
$W_{1,n-1}, n > 7$	1	$\lfloor \frac{2n}{5} \rfloor$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$

Table: Domination parameters of some basic families

Summary

$T = (V, E)$ is a tree other than a path.

▷ A vertex of degree 1 is called a *leaf*.

▷ The number of leafs of T is denoted by $\ell(T)$.

▷ A *major vertex* is a vertex of degree at least 3.

▷ An *exterior major vertex* is any major vertex v such that at least one component of $T - v$ is a path.

▷ The number of exterior major vertices of T is denoted by $ex(T)$.

▷ A *support vertex* is a vertex which is adjacent to a leaf.

▷ The number of support vertices of T is denoted by $su(T)$.

▷ A *strong support vertex* is a vertex which is adjacent to at least two leafs.

Theorem

$\beta(T) = \ell(T) - ex(T)$

$\eta(T) = \gamma(T) + \ell(T) - su(T) = \gamma(T) + \beta(T) - [su(T) - ex(T)]$

$\eta(T) = \gamma(T)$ *iff T has no strong support vertices.*

$\eta(T) = \beta(T) + \gamma(T)$ *iff every support vertex of T is strong.*

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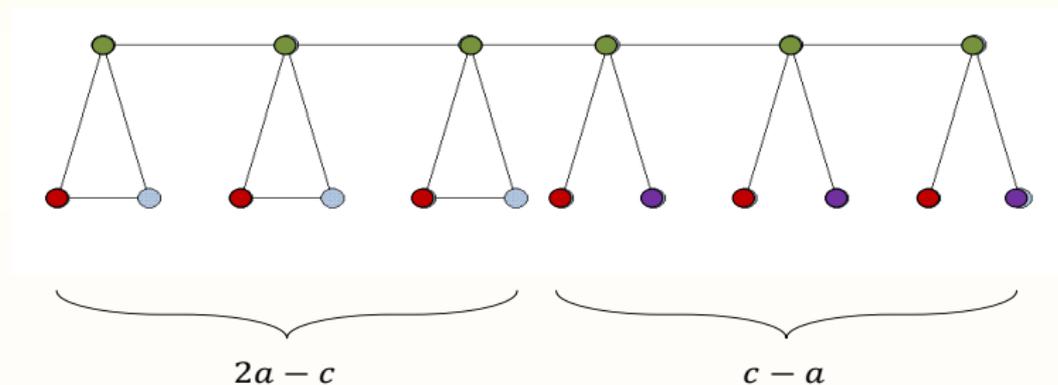# Summary

Theorem

Given three positive integers a, b, c verifying that max$\{a, b\} \leq c \leq a + b$, there always exists a graph G such that $\gamma(G) = a$, $\beta(G) = b$ and $\eta(G) = c$, except for the case $1 = b < a < c = a + 1$.

$2a - c$

$c - a$

Case: $2 \leq a = b < c \leq 2a$

Summary

Theorem

If G is a graph of order n, diameter $D \geq 2$ and location number β, then

$$\beta + D \leq n \leq \left(\left\lfloor \tfrac{2D}{3} \right\rfloor + 1 \right)^{\beta} + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{\beta-1}$$

and both bounds are tight.

Theorem

If G has order n, diameter $D \geq 3$ and metric-location-domination number η, then

$$\eta + \lceil \tfrac{2D}{3} \rceil \leq n \leq \eta + \eta \cdot 3^{\eta-1}$$

Theorem

If G is a graph of order n, diameter $D \geq 2$ and location-domination number λ, then

$$\lambda + \lfloor \tfrac{3D+1}{5} \rfloor \leq n \leq \lambda + 2^{\lambda} - 1$$

Theorem

If G is a graph of order $n \geq 2$, then
(i) $\beta(G) = 1 \Leftrightarrow G = P_n$,
(ii) $\eta(G) = 1 \Leftrightarrow \lambda(G) = 1 \Leftrightarrow G = P_2$

Theorem

If G is a graph of order $n \geq 2$, then
(i) $\beta(G) = n - 1 \Leftrightarrow G = K_n$,
(ii) $\eta(G) = n - 1 \Leftrightarrow \lambda(G) = n - 1 \Leftrightarrow G = K_{1,n-1}$ or $G = K_n$

Moreover, the set of all connected graphs satisfying one of the following conditions has been completely characterized:

 ▷ $\beta(G) = n - 2$, $\beta(G) = n - diam(G)$
 ▷ $\eta(G) = n - 2 \Leftrightarrow \lambda(G) = n - 2$
 ▷ $\eta(G) = 2$
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