Locating domination in graphs[1]

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$G = (V, E)$ is a simple finite connected graph.

- A set D of vertices in G is a *dominating set* if, for every $u \in V(G) \setminus D$:

$$N(u) \cap D \neq \emptyset$$

- The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

- A set $D = \{x_1, \ldots, x_k\}$ is a *locating set* if, for every pair $u, v \in V(G)$,

$$(d(u, x_1), \ldots, d(u, x_k)) \neq (d(v, x_1), \ldots, d(v, x_k)).$$

- The *metric dimension* (also called the *location number*) $\beta(G)$ is the minimum cardinality of a locating set of G.

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- A set D of vertices in a graph G is a *locating dominating set* if it is both locating and dominating.
- The *metric-location-domination number* $\eta(G)$ is the minimum cardinality of a locating dominating set of G.

$\Longrightarrow$ Let $S_1, S_2 \subseteq V(G)$. If S_1 is dominating and S_2 is locating, then $S_1 \cup S_2$ is both locating and dominating. Hence,

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)$$

$\Longrightarrow$ Given three positive integers a, b, c verifying that $\max\{a, b\} \leq c \leq a + b$, there always exists a graph G such that

$$\gamma(G) = a,\ \beta(G) = b \text{ and } \eta(G) = c,$$

except for the case $1 = b < a < c = a + 1$.

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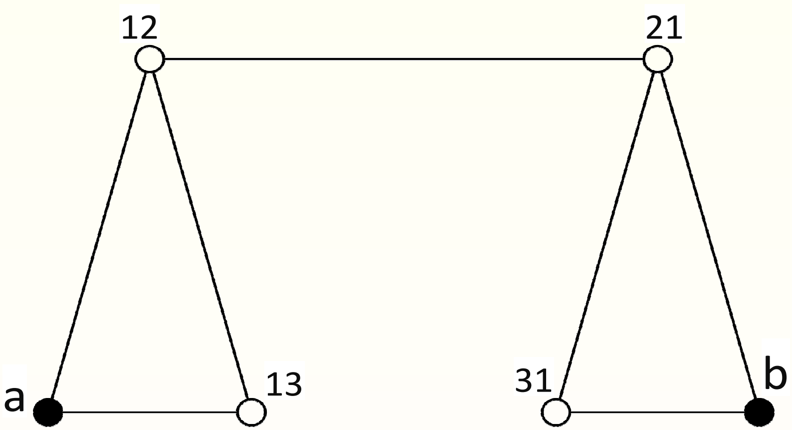
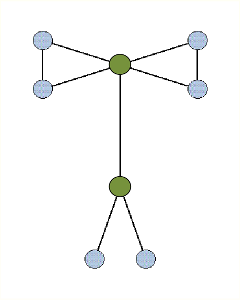
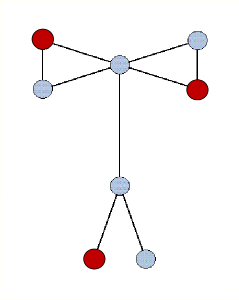
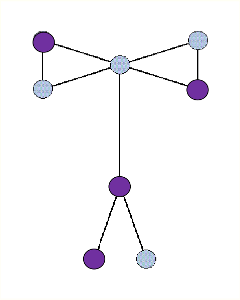


$\eta(G) = 2$, since $\{a, b\}$ is a minimum locating dominating set

		
$\gamma(G) = 2$	$\beta(G) = 3$	$\eta(G) = 4$

In this example: $\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 4 \leq \gamma(G) + \beta(G) = 5$

▷ A set D of vertices in a graph G is a *locating-dominating set* if for every two vertices $u, v \in V(G) \setminus D$,

$$\emptyset \neq N[u] \cap D \neq N[v] \cap D \neq \emptyset.$$

▷ The *location-domination number* $\lambda(G)$ is the minimum cardinality of a locating-dominating set of G.

⟹ Every locating-dominating set is both locating and dominating. Hence,

$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\lambda(G), \gamma(G) + \beta(G)\}$$

and both bounds are tigh.

⟹ Realization theorem? Not yet.

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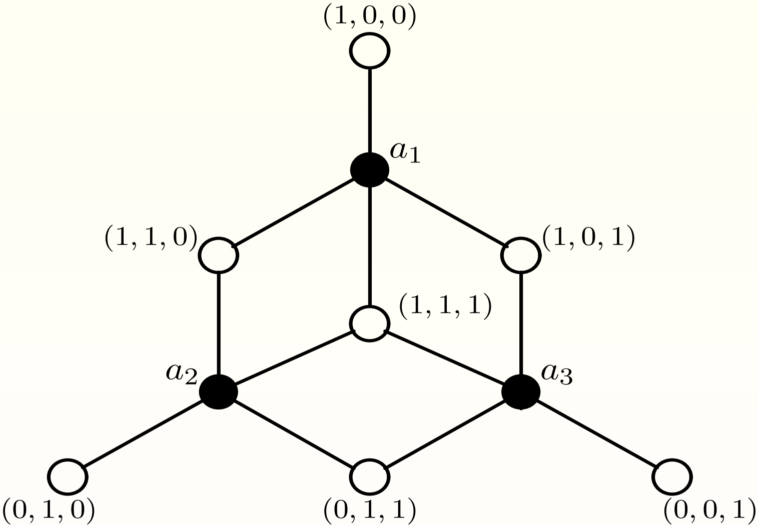
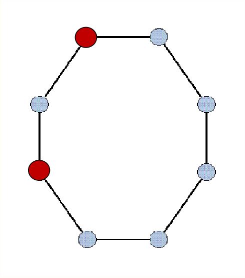
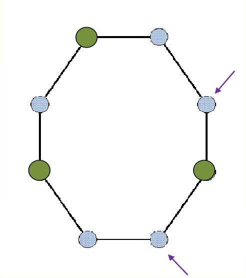
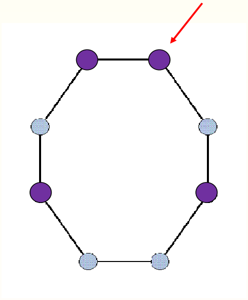


In all cases, digit 0 means "greater than 1"

$\lambda(G) = 3$, since $\{a_1, a_2, a_3\}$ is a minimum locating-dominating set

		
$\beta(G) = 2$	$\gamma(G) = \eta(G) = 3$	$\lambda(G) = 4$

In this example:

$$\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4$$

G	γ	β	η	λ
P_n, $n > 3$	$\lceil \frac{n}{3} \rceil$	1	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$
C_n, $n > 6$	$\lceil \frac{n}{3} \rceil$	2	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{5} \rceil$
K_n, $n > 1$	1	$n-1$	$n-1$	$n-1$
$K_{1,n-1}$, $n > 2$	1	$n-2$	$n-1$	$n-1$
$K_{r,n-r}$, $n-r \geq r > 1$	2	$n-2$	$n-2$	$n-2$
$W_{1,n-1}$, $n > 7$	1	$\lfloor \frac{2n}{5} \rfloor$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$

Domination parameters of some basic families

G is a graph of order n, diameter $D \geq 2$, location number β, metric-location-domination number η and location-domination number λ.

- $\beta + D \leq n \leq \left(\lfloor \frac{2D}{3} \rfloor + 1\right)^{\beta} + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{\beta-1}$
- If $G \neq K_{1,n-1}$, then $\eta + \lceil \frac{2D}{3} \rceil \leq n \leq \eta + \eta \cdot 3^{\eta-1}$
- $\lambda + \lfloor \frac{3D+1}{5} \rfloor \leq n \leq \lambda + 2^{\lambda} - 1$

$*$ In all cases, both bounds are tight.

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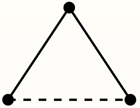
- $\eta(G) = 1 \Leftrightarrow \lambda(G) = 1 \Leftrightarrow G = P_2$

- $\lambda(G) = 2 \Rightarrow \eta(G) = 2$. [converse false]



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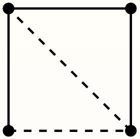

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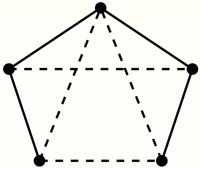
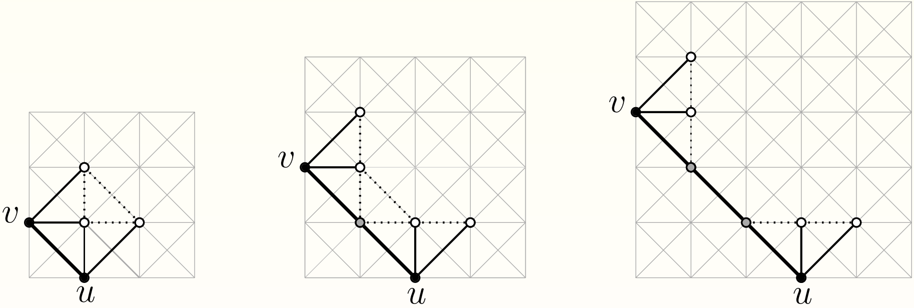


$n = 5$

There are 16 graphs s.t. $\lambda = 2$ (notice that $\lambda = 2 \Rightarrow n \leq 5$)

There are 51 graphs satisfying $\eta = 2$



- ▷ $\eta = 2 \Rightarrow n \leq 8$
- ▷ If $\{u, v\}$ is an η-set, then $d(u, v) \leq 2$.
- ▷ Every graph verifying $\beta \leq 2$ can be embedded into the strong grid.

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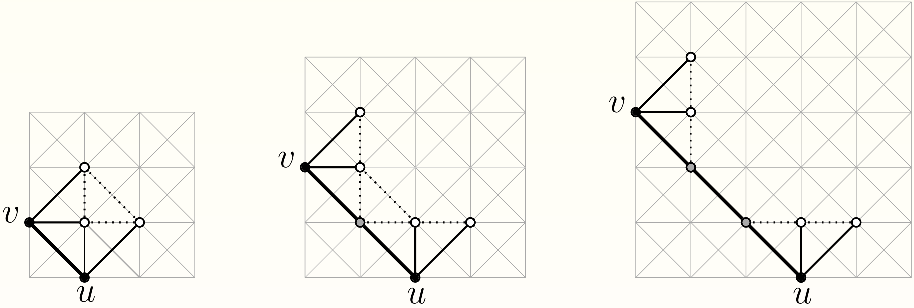

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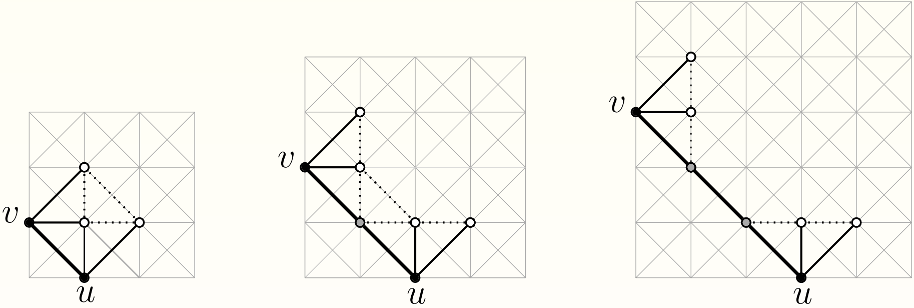

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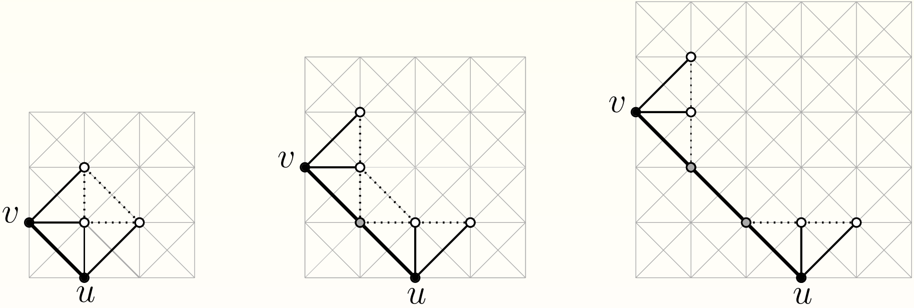

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- $\eta(G) = n - 1 \Leftrightarrow \lambda(G) = n - 1$

- $\lambda(G) = n - 1 \Leftrightarrow G = K_n$ or $G = K_{1,n-1}$

- $\boxed{\lambda(G) = n - 2 \Longleftrightarrow \eta(G) = n - 2}$

- $\lambda(G) = n - 2 \Leftrightarrow G \in F_1 \cup \cdots \cup F_7$, where
  $F_1 = \{K_{r,s} : 2 \leq r \leq s\}$,
  $F_2 = \{K_r + \overline{K}_s : 2 \leq r \leq s\}$, etc.

- $\eta(G) = n - 3 \Rightarrow \lambda(G) = n - 3$ [converse false]

- $\boxed{\text{If } D = 2, \text{ then } \lambda(G) = \eta(G)}$ [for $D \geq 3$, false]

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- $\eta(G) = n - 1 \Leftrightarrow \lambda(G) = n - 1$
- $\lambda(G) = n - 1 \Leftrightarrow G = K_n$ or $G = K_{1,n-1}$
- $\boxed{\lambda(G) = n - 2 \Longleftrightarrow \eta(G) = n - 2}$
- $\lambda(G) = n - 2 \Leftrightarrow G \in F_1 \cup \cdots \cup F_7$, where
  $F_1 = \{K_{r,s} : 2 \leq r \leq s\}$,
  $F_2 = \{K_r + \overline{K}_s : 2 \leq r \leq s\}$, etc.
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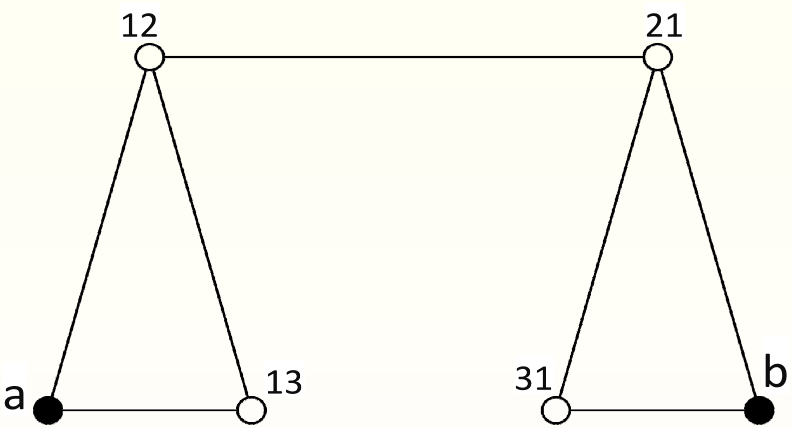

$n = 6$, $D = 3$, $n - 4 = 2 = \eta(G) < \lambda(G) = 3 = n - 3$

- $\eta(K_m \square K_n) = \lambda(K_m \square K_n)$, since $diam(K_m \square K_n) = 2$.
- $\beta(K_m \square K_n) \leq \eta(K_m \square K_n) \leq \beta(K_m \square K_n) + 1$.
- For $m, n \geq 2$, a dominating set S resolves $K_n \square K_m$ iff
  1. there is at most one empty row and at most one empty column;
  2. there is at most one lonely vertex.

$\Longrightarrow$ If $2m - 1 < n$, then $\lambda(K_m \square K_n) = \eta(K_m \square K_n) = \beta(K_m \square K_n) = n - 1$

$\Longrightarrow$ If $m \leq n \leq 2m - 1$, then

$$\lambda(K_n \square K_m) = \begin{cases} \lfloor \frac{2}{3}(n+m-1) \rfloor + 1 & \text{if } n+m = 3k+2 \\ \lfloor \frac{2}{3}(n+m-1) \rfloor & \text{otherwise} \end{cases}$$

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