Global locating domination in bipartite graphs [1]

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[1]Joint work with Carmen Hernando and Mercè Mora.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

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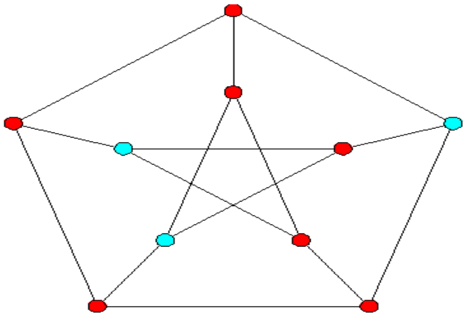

- $\gamma(P) = 3$, since blue vertices form a minimum dominating set.

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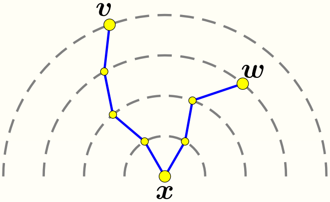

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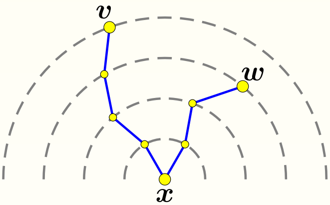

- ▷ A set $S \subseteq V$ is a *locating set* of G if every pair $v, w \in V$ are resolved by some vertex $x \in S$.

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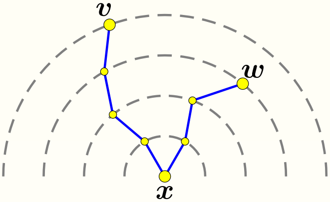

- ▷ A set $S \subseteq V$ is a *locating set* of G if every pair $v, w \in V$ are resolved by some vertex $x \in S$.

- ▷ Let $S = \{u_1, \ldots, u_k\}$ be a locating set. The ordered set:

$$[d(x, u_1), \ldots, d(x, u_k)]$$

is the vector of *metric coordinates* of $x \in V$ with respect to S.

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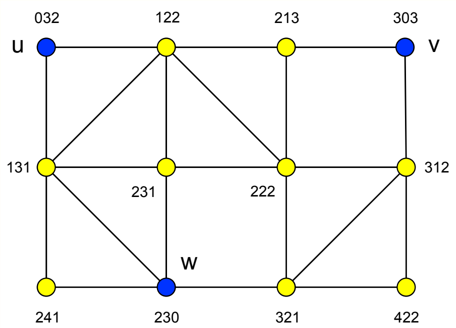
▷ *Metric dimension* of G, $\beta(G)$: cardinality of a metric basis.



- In this graph, $\{u, v, w\}$ is a metric basis.

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$$\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)}$$

- A set D of vertices in a graph G is a *locating-dominating set*, or simply an *LD-set*, if for every two vertices $u, v \in V(G) \setminus D$,

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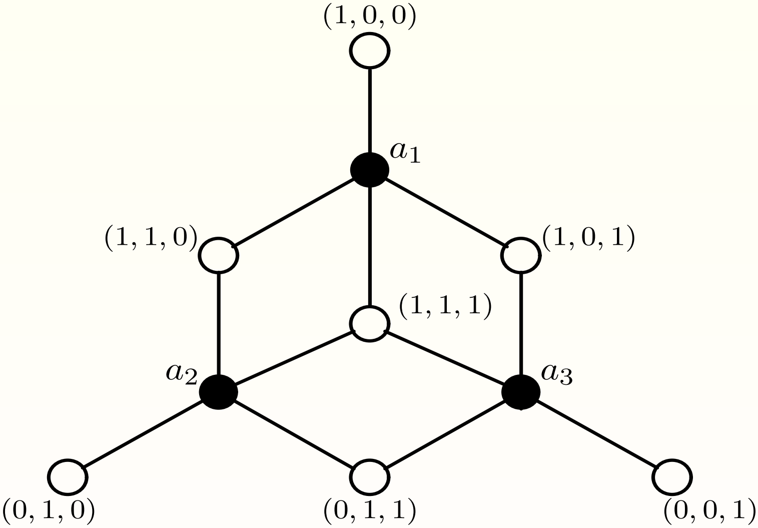
$\Longrightarrow$ Every locating-dominating set is both locating and dominating. Hence,

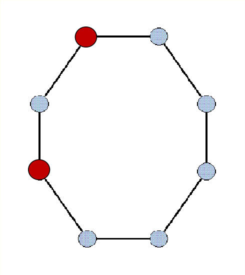
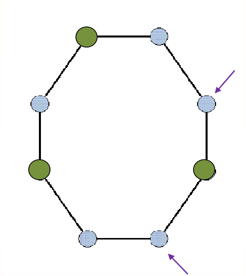
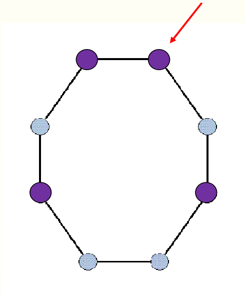
$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\lambda(G), \gamma(G) + \beta(G)\}$$

and both bounds are tight.



In all cases, digit 0 means ”greater than 1”

$\lambda(G) = 3$, since $\{a_1, a_2, a_3\}$ is a λ-code.

		
$\beta(G) = 2$	$\gamma(G) = \eta(G) = 3$	$\lambda(G) = 4$

In this example:

$$\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4$$

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- ▷ Equivalently, an LD-set S of G is called *non-global* if there exists a vertex $w \in V \setminus S$ such that $S \subseteq N(w)$. The vertex w, which is necessarily unique, is called the *dominating vertex* of S.

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⇒ If S is a non-global LD-set of G, then $S + w$ is an LD-set of $\overline{G}$.

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▷ G is of type I: $\boxed{\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)}$

▷ G is of type II: $\boxed{\lambda(\overline{G}) = \lambda(G) + 1}$

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- $\star$ G is disconnected.
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- $\star$ G is disconnected.
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- Moreover, all conditions are tight.

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$\star$ If T is a tree, then the following statements are equivalent:

- $diam(T) = 2$.
- $T \cong K_{1,n-1}$ (i.e., T is a star).
- $\overline{T}$ is disconnected
- $\lambda(\overline{T}) = \lambda(T) = n - 1$.

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- $diam(T) = 3$.
- $T \cong K_2(r, s)$ (i.e., T is a double star).
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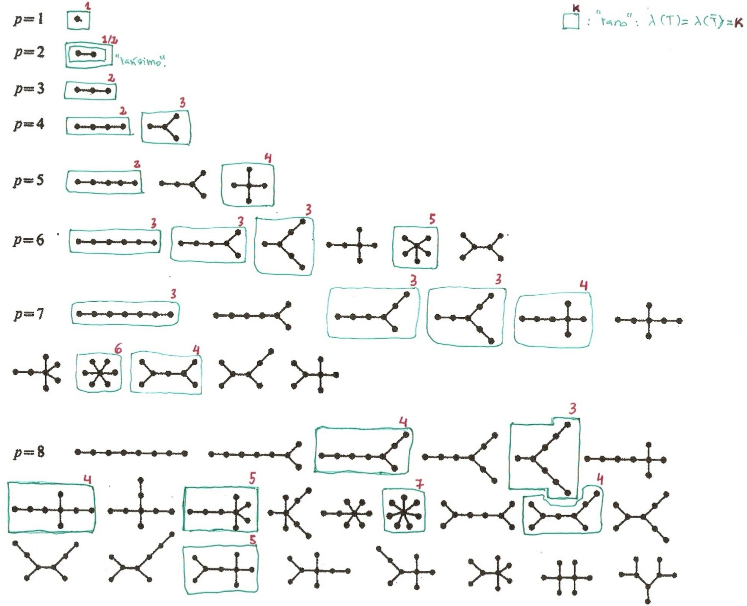
$\star$ $\lambda(\overline{P_{n/2,n/2}}) = \lambda(P_{n/2,n/2}) - 1$, $n \neq 6$, $\lambda(\overline{P_{3,3}}) = \lambda(P_{3,3})$.

There are 48 trees of order at most 8, 23 of them s.t. $\lambda(\overline{T}) = \lambda(T)$.



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- ⋆ Every λ-code of G is non-global.
- ⋆ $rad(G) \leq 2$ and $diam(G) \leq 4$.

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- What about other bipartite graphs?

$\Rightarrow$ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.

⇒ If $G \cong K_2(r, s)$, then $\lambda(\overline{G}) = \lambda(G) - 1$.

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- $\Rightarrow$ If $G \cong K_{r,s}$, then $\lambda(\overline{G}) = \lambda(G)$.
- $\Rightarrow$ If $G \cong C_{2k}$, then $\lambda(G) - 1 \leq \lambda(\overline{G}) \leq \lambda(G)$.

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▷ An (r,s)-graph is a bipartite graph $G = (V = U \cup W, E)$ order $n = r + s$ such that $2 \leq |U| = r \leq |W| = s$.

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$\Rightarrow$ Let S be a λ-code of G. Then,

- If $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$, then S is a global λ-code.
- If $S \cap W = \emptyset$, then $S = U$.
- If $S \cap U = \emptyset$, then $S = W$.

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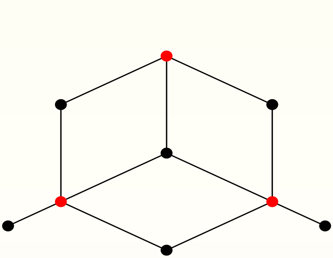
⇒ If $r < s$ and W is a λ-code of G, then $\lambda(\overline{G}) \leq \lambda(G)$.

COROLLARY: If $\lambda(\overline{G}) = \lambda(G) + 1$, then U is the unique λ-code of G.

- $G = (V = U \cup W, E)$, $n = r + s$, with $1 \leq |U| = r \leq |W| = s$.

- $G = (V = U \cup W, E)$, $n = r + s$, with $1 \leq |U| = r \leq |W| = s$.

$\Rightarrow$ If $1 \leq r \leq 3$, then $\lambda(\overline{G}) \leq \lambda(G)$, unless $(r, s) \in \{(3, 6), (3, 7)\}$.



$(r, s) = (3, 6)$ $\quad\quad$ $(r, s) = (3, 7)$

In both cases $\lambda(\overline{G}) = 4 = \lambda(G) + 1$

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$\Rightarrow$ If U is non-global and $2^{r-1} + 2 \leq s$, then $\lambda(\overline{G}) = \lambda(G) + 1$.

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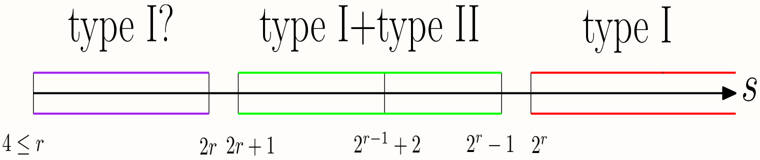

$\Longrightarrow$ **THEOREM:** If $2r+1 \leq s \leq 2^r - 1$, then there exists a bipartite (r,s)-graph $H_{r,s}$ such that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$.

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Sketch of proof:

- G_r is the $(r, 2r+1)$-graph such that $U = \{1, \ldots, r\}$, $W = \{[1],[2],[3],[12],[13],[23],[24],[34],\ldots,[2r],[3r],[12\ldots r]\}$ and $N(j) = \{w \in W : j \in w\}$.

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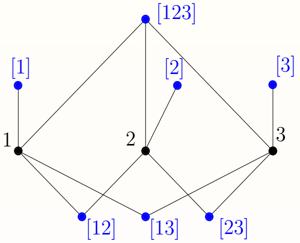
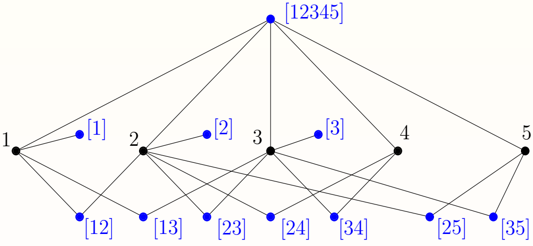


G_3



G_5

$\Longrightarrow$ **THEOREM:** If $2r + 1 \leq s \leq 2^r - 1$, then there exists a bipartite (r, s)-graph $H_{r,s}$ such that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$.

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- G_r is the $(r, 2r+1)$-graph such that $U = \{1, \ldots, r\}$, $W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12 \ldots r]\}$ and $N(j) = \{w \in W : j \in w\}$.
- Prove that U is its unique LD-set, that it is non-global, and that $\lambda(\overline{G}_r) = \lambda(G_r) + 1$.

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- Prove that U is its unique LD-set, that it is non-global, and that $\lambda(\overline{G}_r) = \lambda(G_r) + 1$.
- $H_{r,s}$ is any (r,s)-graph such that $U = \{1, \dots, r\}$, $\{[1], [2], [3], [12], [13], [23], [24], [34], \dots, [2r], [3r], [12 \dots r]\} \subseteq W$, $N(j) = \{w \in W : j \in w\}$ and U is an LD-set.

$\Longrightarrow$ **THEOREM:** If $2r+1 \leq s \leq 2^r - 1$, then there exists a bipartite (r,s)-graph $H_{r,s}$ such that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$.

Sketch of proof:

- G_r is the $(r, 2r+1)$-graph such that $U = \{1, \ldots, r\}$, $W = \{[1], [2], [3], [12], [13], [23], [24], [34], \ldots, [2r], [3r], [12\ldots r]\}$ and $N(j) = \{w \in W : j \in w\}$.

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- Prove that U is its unique LD-set, that it is non-global, and that $\lambda(\overline{H}_{r,s}) = \lambda(H_{r,s}) + 1$.

CONJECTURE 1:

For every bipartite (r, r)-graph G, $\lambda(\overline{G}) \leq \lambda(G)$.

CONJECTURE 2:

For every bipartite (r, s)-graph G, if $r \leq s \leq 2r$, then $\lambda(\overline{G}) \leq \lambda(G)$.